Entropy-conservative and well-balanced discontinuous Galerkin methods for the shallow water equations with uncertainty *

Janina Bender† and Philipp Öffner‡

Abstract.
In this paper, we develop an entropy-conservative discontinuous Galerkin method for the shallow water (SW) equation with random inputs. One of the most popular methods for uncertainty quantification is the generalized Polynomial Chaos approach which we consider in the following manuscript. We apply the stochastic Galerkin (SG) method to the stochastic SW equations. Using the SG approach in the stochastic hyperbolic SW system yields a purely deterministic system which is not necessarily hyperbolic anymore. The lack of hyperbolicity leads to ill-posedness and stability issues in numerical simulations. By transforming the system using Roe variables, hyperbolicity can be ensured and an entropy-entropy flux pair is known from a recent investigation by Gerster and Herty (Commun. Comput. Phys., 27, pp. 639–671, 2020). We use this pair and determine a corresponding entropy flux potential. Then, we construct entropy conservative numerical two-point fluxes for this augmented system. By applying these new numerical fluxes in a nodal discontinuous Galerkin spectral element method with flux differencing ansatz, we obtain a provable entropy conservative (dissipative) and well-balanced scheme. In numerical experiments, we validate our theoretical findings.

Key words. Shallow water equations; entropy conservation/dissipation; uncertainty quantification; discontinuous Galerkin; generalized Polynomial Chaos; well-balanced

AMS subject classifications (2020). 65M12, 65M60, 65M70

1. Introduction.

Many problems in natural sciences and engineering are modelled by hyperbolic balance laws. One such system are the shallow water (SW) equations used in many geophysical processes such as river flows or coastal areas but can be also applied to atmospheric flows. The application of the SW is widely and the development of effective and accurate numerical methods for the SW equations has received much interest in the last decades, and it is still ongoing, cf. [9, 10, 33, 34, 45, 58] and references therein. In particular, the construction of entropy (energy) conservative (dissipative) methods has been of great interest [18, 23, 32, 43, 53]. Contrarily, in many real applications and models real data are applied which comes with uncertainties due to empirical approximations or measuring errors resulting in a stochastic partial differential system. In the context of hyperbolic conservation/balance laws, uncertainties can appear in the source terms, initial or boundary data or even the fluxes and different approaches exist to solve such stochastic PDE systems. Classical techniques are either stochastic collocation (SC), Monte Carlo (MC) algorithms or the generalized Polynomial Chaos (gPC) approach (using a stochastic Galerkin (SG) ansatz), cf. [35, 40, 39, 14, 52, 60, 59, 48]. All of them (SC, MC or SG) come with some advantages and disadvantages, where a nice summary in the context of hyperbolic equations can be found in [2]. In this manuscript, we focus on the gPC(SG) approach in the context of the SW equations with uncertainties. The SG ansatz applied to stochastic hyperbolic equations yields an augmented purely deterministic system, denoted as SG system in the following, where classical solvers (MUSCL, FV, DG) can be used. However, a drawback of this approach is that one may lose the hyperbolicity of the SG system [15] and classical solvers in general fail to solve the extended

*February 27, 2023
Corresponding author: Philipp Öffner
Funding: PÖ was supported by the Gutenberg Research College, JGU Mainz.
†Institute of Mathematics, University of Kassel, Kassel, Germany, (jmbender@mathematik.uni-kassel.de, orcid.org/0009-0007-3938-6800)
‡Institute of Mathematics, Johannes Gutenberg University, Mainz, Germany, (poeffner@uni-mainz.de, orcid.org/0000-0002-1367-1917)

system due to its ill-posedness. Further, the question rises about what kind of underlying structures we want to preserve, like what kind of entropies we have. To overcome the loss of hyperbolicity, different techniques have been already proposed. Inspired by kinetic theory, the entropy closure methods [42] or filtering procedures to ensure the hyperbolicity [16, 46] have been developed. Alternatively, it was demonstrated in [61] that by a pseudo-spectral approach with suitable quadrature rules, one can rewrite the SG scheme as an SC scheme on a set of specific nodes. The collocation scheme preserves the hyperbolicity of the original hyperbolic system. Also, recently, it was demonstrated in [13, 12] that only a finite collection of positivity conditions on the stochastic water height at selected quadrature points in parameter space is enough to preserve the hyperbolicity in the SG system. Alternatively, by introducing Roe variables directly inside the Galerkin projections one can as well ensure the hyperbolicity of the SG system [40, 41] but the question about the definition and existence of entropies rises. In recent works [24, 25] the authors were finally able to determine entropies for several hyperbolic SG systems, in particular the SW system (SG-SW). In the following manuscript, we use this new development and construct a high-order entropy conservative DG method for the SG-SW system. Different from recent paper [61], our approach is truly intrusive, by which we mean that all integrals of the Galerkin ansatz are exactly computed in a pre-computation step and not during a simulation. We apply the flux differencing [7, 23, 43] ansatz in our DG formulation. The main idea is to use a split formulation inside the discretization, where the splitting is determined by entropy conservative numerical fluxes (in the sense of Tadmor [50]). To determine entropy conservative numerical fluxes, the flux potential is needed, and we derive for our entropy-entropy flux pair form [24] a flux potential. We use this potential finally to develop, in an analogous manner as in [37], an entropy conservative, numerical flux for our SG-SW system. Our result yields a new reliable numerical method for shallow water equations with uncertainties. The paper is organized as follows. In Section 2, the numerical framework is introduced. We focus on the discontinuous Galerkin spectral element (DGSEM) method using a flux differencing approach. If entropy conservative fluxes are applied inside the discretization, we obtain a provable entropy conservative (dissipative) scheme. We also introduce in this section the gPC approach. Afterwards, in Section 3 the shallow water equation with uncertainty is introduced together with the Roe variable transformation, and the SG approach is used to determine the augmented system and derive the entropy-entropy flux pair following [24]. In Section 4, we derive first the entropy flux potential for the entropy-entropy flux pair and use it to develop entropy conservative numerical fluxes for the augmented SG-SW system. In numerical simulations in Section 5, we give a proof of concept that the application of such fluxes inside the DGSEM-flux differencing method results in an entropy conservative and well-balanced high-order scheme. A summary with an outlook finishes the main part of the manuscript where in the Appendix A the concrete formulas of the flux potentials and numerical fluxes can be found up to the second order series expansion of the SG approach together with some basic properties of the Haar wavelet expansions. By slide modifications, we explain also how to include bottom topography inside our study, and we specify an entropy conservative numerical flux with bottom topography for completeness.

2. Numerical framework.

In the following section, we shortly repeat the numerical framework which we use inside the manuscript. In particular, we introduced the discontinuous Galerkin spectral element method (DGSEM) and explain how we can ensure entropy conservation (dissipation) for the scheme. Afterwards, we shortly introduce the gPC approach. This framework is used to extend the shallow water equations in the next section for which we will finally construct an entropy conservative (dissipative) method.

2.1. Discontinuous Galerkin spectral element method.

There exist various approaches to construct entropy conservative (dissipative) numerical methods for hyperbolic conservation/balance laws, cf. [1, 3, 4, 8, 21, 31, 19, 17, 20, 38, 30, 36] and references therein. In our work, we focus on the discontinuous Galerkin method with flux differencing ansatz as described in [7, 22] and used in the SW context in [23, 43, 55, 53, 56]. To clarify the method we start with a nodal DG scheme using summation-by-parts (SBP) operators and consider for simplicity the hyperbolic conservation law

$$\partial_t u(t,x) + \partial_x f(u(t,x)) = 0.$$

As usual in finite elements, we make a domain decomposition $x_{1/2} < x_{3/2} < \cdots < x_{N+1/2}$, $\Omega_i = [x_{i-1/2}, x_{i+1/2}], \Delta x_i = x_{i+1/2} - x_{i-1/2}$. Instead of calculating everything in Ω_i, every element is mapped into the standard element $[-1,1]$ where all the calculations are done.
In classical DG, we multiply with a test function $v^h \in \mathcal{V}^h$ where $\mathcal{V}^h$ is our solution space (broken polynomial space) and integrate in space. Then, we seek $u^h \in \mathcal{V}^h$ such that for each $v^h \in \mathcal{V}^h$ and $1 \le i \le N$:

$$\int_{\Omega_i} \frac{\partial u^h}{\partial t} v^h \mathrm{d}x - \int_{\Omega_i} f(u^h) \frac{\partial v^h}{\partial x} \mathrm{d}x = f^{\text{num}}_{i-1/2} v^h(x^+_{i-1/2}) - f^{\text{num}}_{i+1/2} v^h(x^-_{i+1/2}), \tag{2.1}$$

where we used integration-by-parts once to shift the derivative from the flux function to the test function. The right side corresponds to the boundary values. Since the flux functions at the boundaries are not unique, numerical fluxes have to be applied. Further, the test function v^h is evaluated also at the boundary where the superscript $\pm$ denotes the left or right value between two neighbouring elements specifying which approximations have to be used.
Instead working with (2.1), we use integration-by-parts again and obtain the strong form

$$\begin{aligned}\int_{\Omega_i} \left(\frac{\partial u^h}{\partial t} + \frac{\partial f(u^h)}{\partial x} \right) v^h \mathrm{d}x = &\left(f(u^h(t, x^-_{i+1/2})) - f^{\text{num}}_{i+1/2} \right) v^h(x^-_{i+1/2}) \\ &- \left(f(u^h(t, x^+_{i-1/2})) - f^{\text{num}}_{i-1/2} \right) v^h(x^+_{i-1/2}).\end{aligned}$$

Focusing now on the reference element, instead of evaluating the integrals exactly, we apply the Gauss-Lobatto quadrature $-1 = \xi_0 < \xi_1 < \cdots < \xi_p = 1$ with corresponding quadrature weights $\{\omega_j\}_{j=0}^p$. Further, we use a Lagrangian nodal basis for these quadrature points, i.e. $L_j(\xi_l) = \delta_{jl}$, and define the discrete inner $\langle u, v \rangle_\omega = \sum_{j=0}^{p} \omega_j u(\xi_j) v(\xi_j)$. To make the connection to the SBP operators, we have

- Difference matrix $\underline{\underline{D}}$ with $\underline{\underline{D}}_{jl} = L'_l(\xi_j)$
- Mass matrix $\underline{\underline{M}}_{jl} = \langle L_j, L_l \rangle_\omega = \omega_j \delta_{jl}$, so that $\underline{\underline{M}} = \operatorname{diag}\{\omega_0, \dots, \omega_p\}$
- Stiffness matrix $\underline{\underline{Q}}_{jl} = \left\langle L'_j, L_l \right\rangle_\omega = \langle L_j, L'_l \rangle_\omega$, Boundary matrix $\underline{\underline{B}} = \operatorname{diag}(-1, 0, \dots, 0, 1)$

The above operators fulfil the SBP property as shown for instance in [7, Theorem 3.1]. In our nodal DG formulation, we approximate the flux function as well by a polynomial and since v^h has been arbitrary, we get the strong DG scheme for one node

$$\frac{\Delta x}{2} \partial_t u_j + \sum_{l=0}^{p} D_{jl} f_l = \frac{\tau_j}{w_j} (f_j - f^{\text{num}}_j) \tag{2.2}$$

where $\tau_j \in \{-1, 0, 1\}$ depending which node is considered. Finally, a system of equations is obtained. Method (2.2) is referred to as the discontinuous Galerkin spectral element method. However, in general, this method is not entropy conservative (dissipative). To ensure entropy conservation, we have to modify the scheme and use suitable numerical fluxes. Therefore, we assume that the entropy function η is strictly convex, defining the entropy variable $v = \eta'(u) =: \partial_u \eta(u)$. Due to the strictly convex entropy function η, there exists as well a potential $\psi'(v) = f(u(v)$. A consistent, symmetric two-point numerical flux $f^{\text{num}}(u_l, u_r)$ is called entropy conservative (dissipative) (in the sense of Tadmor) if it satisfies

$$[[v]] f^{\text{num}}(u_l, u_r) \overset{(\leq)}{=} [[\psi]] \iff (v_r - v_l) f^{\text{num}}(u_l, u_r) \overset{(\leq)}{=} (\psi_r - \psi_l). \tag{2.3}$$

Here, v_l, v_r and ψ_l, ψ_r are the entropy variables and potential at the left and right states, whereas $[[\cdot]]$ denotes the jump, cf. [50, 51] for more details. Combing back to the DG formulation to balance the internal entropy, we use a split formulation. The scheme reads than

$$\frac{\Delta x}{2} \partial_t u_j + 2 \sum_{l=0}^{p} D_{jl} f_S^{\text{num}}(u_j, u_l) = \frac{\tau_j}{w_j} (f_j - f_j^{\text{num}}), \tag{2.4}$$

where f_S^{num} are numerical two-point fluxes. Finally, the properties of the scheme depend highly on the selected fluxes as the following theorem describes, cf. [7, Section 3].

Theorem 2.1 (Flux Differencing Theorem). *If $f_S^{\text{num}}(u_j, u_l)$ is consistent and symmetric, then (2.4) is conservative and high-order accurate. If we further assume that $f_S^{\text{num}}(u_j, u_l)$ is entropy conservative (2.3), then (2.4) is also entropy conservative within a single element. If f^{num} is further entropy conservative (dissipative), then the resulting scheme is entropy conservative (dissipative) in general.*

Because of this theorem, it is enough to construct consistent, symmetric and entropy-conservative fluxes to develop a high-order entropy-conservative scheme. This will be the major part in Section 4.

Remark 2.2. The considerations can be extended to multi-dimensional space dimensions by using a tensor-product strategy. Additionally, it is possible to get diagonal operators for triangle grids if enough nodes are added at the boundaries.

To obtain an entropy dissipative scheme, an entropy dissipative flux has to be used at the element interfaces. By adding artificial dissipation (in the local Lax-Friedrich sense) to our entropy conservative flux, we would obtain an entropy dissipative flux and use them for f^{num} an entropy dissipative scheme, cf. [37].

2.2. Polynomial chaos for hyperbolic system.

We introduce the *generalized polynomial chaos* (gPC) expansion referring to [24, 40, 37], which aims to express the stochastic problem in a deterministic surrounding and use classical numerical schemes. Therefore, let $(\Omega, \mathbb{F}, \mathbb{P})$ be a probability space with event space Ω, and probability measure $\mathbb{P}$ defined on the σ-field $\mathbb{F}$. We consider $\xi = \{\xi_k(\omega)\}_{j=0}^N$ the set of N independent and identically distributed random variables for $\omega \in \Omega$. Based on that, we introduce the function space $\mathbb{L}^2(\Omega, \mathbb{P}) := \{Z | Z : \Omega \to \mathbb{R} \quad \text{mesurable}, \quad \|Z\| < \infty\}$ with scalar product

$$\langle Z_1 Z_2 \rangle := \int Z_1 Z_2 \mathrm{d}\mathbb{P}. \tag{2.5}$$

We denote by $\{\phi_k(\xi)\}_{k=0}^\infty$ for $\mathbb{L}^2(\Omega, \mathbb{P})$ a set of orthogonal basis functions. Focusing on our conservation law (2.1), we are interested in quantities u which now are not only dependent on space x and time t but also influenced by the random variable ξ, i.e. $u(t, x, \xi)$ is searched. We proceed analogously to the deterministic case and expand the weak formulation by $u(t, x; \xi)$. This leads us to

$$\int_0^T \int_{\mathbb{R}} \mathbb{E}\Big[\Big\{u(t, x; \xi) \cdot \frac{\partial \varphi}{\partial t} + f(u(t, x; \xi)) \cdot \frac{\partial \varphi}{\partial x}\Big\} \phi_k(\xi)\Big] \mathrm{d}x \mathrm{d}t = 0,$$

where the expected value with corresponding ϕ_k contains the integration over ξ i.e.

$$\mathbb{E}[u(x, t; \xi)] = \int_\Omega u(x, t; \xi(\omega)) \mathrm{d}\mathbb{P}(\omega) = \int_{\mathbb{R}} u(x, t; y) \rho(y) \mathrm{d}y.$$

Here, ρ denotes the probability density function. As mentioned in [2, 54], the foundation of gPC is to use the spectral expansion

$$u(x,t;\xi) = \sum_{k=0}^{\infty} u_k(t,x)\phi_k, \tag{2.6}$$

for a random field. The set of coefficients $\{u_k(x,t)\}$ are purely deterministic and have to be calculated using our favourite numerical method. Now the *generalized polynomial chaos* (gPC) is introduced as a set of orthogonal subspaces $\hat{S}_k \subset \mathbb{L}^2(\Omega,\mathbb{P})$ with $S_K := \bigoplus_{k=0}^{K} \hat{S}_k \to \mathbb{L}^2(\Omega,\mathbb{P})$ for $K \to \infty$. We consider now the projection operator onto the gPC basis of degree $K \in \mathbb{N}_0$. It is

$$G_K[u](t,x;\xi) := \sum_{k=0}^{K} \hat{u}_k(t,x)\phi_k(\xi), \quad \hat{u}_k(t,x) := \frac{\langle u(t,x;\cdot)\phi_k(\cdot)\rangle}{\|\phi_k\|^2}, \tag{2.7}$$

which approximates for any fixed (t,x) the solution. The convergence of G_K to u for $K \to \infty$ is ensured by the Cameron-Martin theorem [6], i.e. it guarantees, that the truncated series expansion (2.7) converges to the spectral expansion (2.6). Later, we calculate $\hat{u}_k$ using our spectral Galerkin ansatz. Through straightforward calculations, we have for the first two moments using an orthogonal basis:

- Expected value

$$\mathbb{E}\left[G_K[u]\right](t,x) = \hat{u}_0(x,t)\int_{\mathbb{R}} \phi_0(y)\rho(y)\mathrm{d}y + \int_{\mathbb{R}} \sum_{k=1}^{K} \hat{u}_k(x,t)\phi_k(y)\rho(y)\mathrm{d}y = \hat{u}_0(x,t)$$

- Variance

$$\mathrm{Var}\left[G_K[u]\right](t,x) = \mathbb{E}[\hat{u}^2(x,t;\cdot)] - \mathbb{E}^2[\hat{u}(x,t;\cdot] = \sum_{k=1}^{K} \hat{u}_k^2(x,t)\mathbb{E}[\phi_k^2] = \sum_{k=1}^{K} \hat{u}_k^2(x,t)$$

Additionally, we assume for simplicity normed basis functions, i.e. $\|\phi_k\| = 1$. For our later considerations, we need the following operator, called **Galerkin product**. It is given by

$$\hat{G}_K[y,z](t,x;\xi) := \sum_{k=0}^{K} (\hat{y} * \hat{z})_k(t,x)\phi_k(\xi) \tag{2.8}$$

with $(\hat{y} * \hat{z})_k(t,x) := \sum_{i,j=0}^{K} \hat{y}_i(t,x)\hat{z}_j(t,x)\langle\phi_i\phi_j\phi_k\rangle$. This can be reformulated using the symmetric matrix $\mathcal{P}(\hat{y}) := \sum_{l=0}^{K} \hat{y}_l M_l^K$ with $M_l^K := \left(\langle\phi_l\phi_i\phi_j\rangle\right)_{i,j=0,\dots,K}$. We obtain $\hat{y} * \hat{z} = \mathcal{P}(\hat{y})\hat{z}$ and $\mathcal{P} \in \mathbb{R}^{(K+1)\times(K+1)}$. The product in M is derived from the scalar product (2.5) and reads as $\langle\phi_l\phi_i\phi_j\rangle = \int_\Omega \phi_l(\xi)\phi_i(\xi)\phi_j(\xi)\mathrm{d}\mathbb{P}$. In our further computations, we reduce the set

$$\mathbb{H}^+ := \left\{\hat{u} \in \mathbb{R}^{K+1} | \mathcal{P}(\hat{u}) \text{ is strictly positive definite}\right\}. \tag{2.9}$$

This condition is fulfilled if we have a positive expansion [27, 49]. We will not introduce that technically, but explain what it means for our model in Section 3. We need to make an additional note on some properties of the matrix $\mathcal{P}(\hat{\alpha})$ from [24]:

Lemma 1. *The following properties are equivalent for the matrix $\mathcal{P}(\alpha)$, which is defined as above:*
1. *The precomputed matrices $\mathcal{M}_l$ and $\mathcal{M}_k$ commute for all $l, k = 0, \dots, K$.*
2. *The matrices $\mathcal{P}(\hat{\alpha})$ and $\mathcal{P}(\hat{\beta})$ commute for all $\hat{\alpha}, \hat{\beta} \in \mathbb{R}^{K+1}$.*
3. *There is an eigenvalue decomposition $\mathcal{P}(\hat{\alpha}) = V D_{\mathcal{P}}(\hat{\alpha}) V^T$ with constant eigenvectors.*

These properties are important but not universal. We call bases that satisfy Lemma 1 Haar type bases. Those are rare and have to be well-chosen. The Haar-wavelets used later are such a basis, whereas classical orthogonal polynomials are not. Finally, we fix the following property of $\mathcal{P}(\hat{\alpha})$ which is needed later.

Lemma 2. *Property of $\mathcal{P}(\hat{\alpha})$. For any $\hat{\alpha}, \hat{\beta}, \hat{\gamma} \in \mathbb{R}^{K+1}$, it holds $\hat{\alpha}^T \mathcal{P}(\hat{\gamma}) \hat{\beta} = \hat{\beta}^T \mathcal{P}(\hat{\gamma}) \hat{\alpha}$.*

Proof. We consider the transposition $\left(\hat{\alpha}^T \mathcal{P}(\hat{\gamma}) \hat{\beta}\right)^T = \hat{\beta}^T \mathcal{P}(\hat{\gamma})^T \hat{\alpha} = \hat{\beta}^T \mathcal{P}(\hat{\gamma}) \hat{\alpha}$. The last equality holds due to the symmetry of $\mathcal{P}(\hat{\gamma})$. The assertion follows because $\hat{\alpha}^T \mathcal{P}(\hat{\gamma}) \hat{\beta}$ is a scalar. ∎

3. Shallow water equations with uncertainties.

In the following section, we shortly repeat the SW model with uncertainty together with Roe variable transformation and introduce the entropy-entropy flux pair from [24]. The SW system can be derived from the Navier-Stokes equations and is defined in the one-dimensional case via

$$\frac{\partial}{\partial t}\begin{pmatrix} h \\ h\mathsf{v} \end{pmatrix} + \frac{\partial}{\partial x}\begin{pmatrix} h\mathsf{v} \\ h\mathsf{v}^2 + \frac{1}{2}gh^2 \end{pmatrix} = \begin{pmatrix} 0 \\ gh\frac{\partial}{\partial x}b \end{pmatrix}, \tag{3.1}$$

where h denotes water high, v the velocity, g the gravitational constant and b the bottom topography. The first equation describes the balance of mass and the second one the balance of momentum $q = h\mathsf{v}$. The conserved vector is $u = (h, q)^T$. The right side denotes the source terms (here only bottom topography) but could also include Coriolis force and friction. For simplicity, we assume flat bathymetry ($b \equiv 0$) in the following. Instead of having a purely deterministic system (3.1), we have an additional random input ξ and the stochastic SW reads

$$\frac{\partial}{\partial t}\begin{pmatrix} h(t,x;\xi) \\ q(t,x;\xi) \end{pmatrix} + \frac{\partial}{\partial x}\begin{pmatrix} q(t,x;\xi) \\ \frac{q^2(t,x;\xi)}{h(t,x;\xi)} + \frac{1}{2}gh^2(t,x;\xi) \end{pmatrix} = 0. \tag{3.2}$$

As demonstrated in [15], by using an SG extension of (3.2) we lose the hyperbolicity of the system. Therefore, we introduce first the Roe variable transformation for the SWE.

Definition 3. *(Roe variables). With velocity $\mathsf{v}(u) := \frac{q}{h}$ as auxiliary variables the Roe variables are defined as $\omega := (\alpha, \beta) := \left(\sqrt{h}, \sqrt{h}\mathsf{v}(y)\right)$ and the gPC modes as $\hat{\omega} := (\hat{\alpha}, \hat{\beta})$ for $\hat{\alpha} \in \mathbb{H}^+$ defined on the set*

$$\mathbb{H}^+ = \{\hat{\alpha} \in \mathbb{R}^{K+1} | \mathcal{P}(\hat{\alpha}) \text{ is strictly positive definite}\}.$$

The mapping between Roe and conserved variables is

$$\mathcal{Y} : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}^+ \times \mathbb{R}, \quad \hat{\omega} \mapsto \begin{pmatrix} \alpha^2 \\ \alpha\beta \end{pmatrix} = u \quad \text{for} \quad K = 0,$$

$$\hat{\mathcal{Y}} : \mathbb{H}^+ \times \mathbb{R}^{K+1} \to \left(\mathbb{R}^+ \times \mathbb{R}^K\right) \times \mathbb{R}^{K+1}, \quad \hat{\omega} \mapsto \begin{pmatrix} \hat{\alpha} * \hat{\alpha} \\ \hat{\alpha} * \hat{\beta} \end{pmatrix} = \hat{u} \quad \text{for} \quad K \in \mathbb{N}.$$

So, all predefinitions are set to apply the stochastic Galerkin formulation with Roe variable transform on the SW equations. A formulation of SW in terms of Roe variables is given through

$$\frac{\partial}{\partial t}\begin{pmatrix}\alpha^2(t,x;\xi)\\ \alpha\beta(t,x;\xi)\end{pmatrix}+\frac{\partial}{\partial x}\begin{pmatrix}(\alpha\beta)(t,x;\xi)\\ \beta^2(t,x;\xi)+\frac{1}{2}g\alpha^4(t,x;\xi)\end{pmatrix}=0.$$

The SG expansion leads us to

$$\left\langle\frac{\partial}{\partial t}\begin{pmatrix}\hat{\mathcal{G}}_K[\alpha,\alpha](t,x;\xi)\\ \hat{\mathcal{G}}_K[\alpha,\beta](t,x;\xi)\end{pmatrix}+\frac{\partial}{\partial x}\begin{pmatrix}\hat{\mathcal{G}}_K[\alpha,\beta](t,x;\xi)\\ \hat{\mathcal{G}}_K[\beta,\beta](t,x;\xi)+\frac{1}{2}g\hat{\mathcal{G}}_K^4[\alpha](t,x;\xi)\end{pmatrix},\phi_k(\xi)\right\rangle=0, \tag{3.3}$$

where $\mathcal{G}$ denotes the Galerkin product (2.8). We need the Roe variable formulation to ensure hyperbolicity. We work with the gPC modes of (3.3) and obtain

$$\begin{pmatrix}\hat\alpha*\hat\alpha\\ \hat\alpha*\hat\beta\end{pmatrix}_t+\begin{pmatrix}\hat\alpha*\hat\beta\\ \hat\beta*\hat\beta+\frac{1}{2}g(\hat\alpha*\hat\alpha)*(\hat\alpha*\hat\alpha)\end{pmatrix}_x=0$$

with the flux function $\hat f(\hat u):=\hat f_1(\hat u)+\hat f_2(\hat u)$ formulated in conservative variables $\hat u=\hat{\mathcal{Y}}(\hat\omega)$ for

$$\hat f_1(\hat u):=\begin{pmatrix}\hat q\\ \frac{1}{2}g\hat h*\hat h\end{pmatrix}\quad\text{and}\quad \hat f_2(\hat u):=\tilde f\left(\hat{\mathcal{Y}}^{-1}(\hat u)\right):=\begin{pmatrix}0\\ \hat\beta*\hat\beta\end{pmatrix}. \tag{3.4}$$

To express the entropy-entropy flux pair and to derive later the potential and the numerical fluxes, we introduce some additional variables to express the expanded system similar to the original one in conservative variables. Further, we introduce matrices to simplify the notation which will be useful due to some technical parts in the proofs later.

Definition 4. *Similar to the velocity in the deterministic case, we define* $\hat{\mathrm{v}}(\hat\omega):=\mathcal{P}^{-1}(\hat\alpha)\hat\beta$, $\hat{\mathrm{v}}^2(\hat\omega):=\mathcal{P}_2(\hat\omega)\hat\beta$ *with the matrices representation* $\mathcal{P}_1(\hat\omega):=\mathcal{P}(\hat\beta)\mathcal{P}^{-1}(\hat\alpha)$ and $\mathcal{P}_2(\hat\omega):=\mathcal{P}(\hat\beta)\mathcal{P}^{-2}(\hat\alpha)$.

Finally, we have all the ingredients to repeat the main result of [24] the definition of an entropy-entropy flux pair for SW equations.

Theorem 5 (Shallow Water Equations [24]). *Let a Haar type expansion be given, and let states in the open, admissible set*

$$\mathbb{H}=\left\{\hat u:=(\hat h,\hat q)^T\in\left(\mathbb{R}^+\times\mathbb{R}^K\right)\times\mathbb{R}^{K+1}|\hat\alpha\in\mathbb{H}^+\quad\text{for}\quad(\hat\alpha,\hat\beta)^T=\hat{\mathcal{Y}}^{-1}(\hat u)\right\}$$

be given. Then, the Jacobian of the flux function (3.4) *is*

$$\mathrm{D}_{\hat u}\hat f(\hat u)=\begin{pmatrix}\mathbf{0}&\mathbf{1}\\ g\mathcal{P}(\hat h)-\mathcal{P}_1^2(\hat\omega)&2\mathcal{P}_1(\hat\omega)\end{pmatrix}$$

for $\hat\omega=(\hat\alpha,\hat\beta)$ *and* $\mathcal{P}_1(\hat\omega)=\mathcal{P}(\hat\beta)\mathcal{P}^{-1}(\hat\alpha)$. *The eigenvalue decomposition* $\mathrm{D}_{\hat u}\hat f(\hat u):=[\mathcal{V}\hat T(\hat\omega)]\hat\Lambda(\hat\omega)[\mathcal{V}\hat T(\hat\omega)]^{-1}$ *reads as*

$$\hat\Lambda^{\pm}(\hat\omega):=D_{\mathcal{P}}(\hat\beta)D_{\mathcal{P}}^{-1}(\hat\alpha)\pm\sqrt{gD_{\mathcal{P}}(\hat h)},\quad \hat\Lambda(\hat\omega):=\operatorname{diag}\{\hat\Lambda^+(\hat\omega),\hat\Lambda^-(\hat\omega)\},$$

$$\hat T(\hat\omega):=\begin{pmatrix}\mathbf{1}&\mathbf{1}\\ \hat\Lambda^+(\hat\omega)&\hat\Lambda^-(\hat\omega)\end{pmatrix},\qquad \mathcal{V}:=\operatorname{diag}\{V,V\},$$

while $D_{\mathcal{P}}$ results of the eigenvalue decomposition of $\mathcal{P}$ in Lemma 1. As entropy-entropy flux pair we obtain $(\eta, \mu) := (\eta_1 + \eta_2, \mu_1 + \mu_2)\,(\hat{u})$ with

$$\begin{aligned} \eta_1(\hat{u}) &:= \frac{g}{2}\left\|\hat{h}\right\|_2^2 \quad \text{and} \quad \eta_2(\hat{u}) := \tilde{\eta}_2\left(\hat{\mathcal{Y}}^{-1}(\hat{u})\right) := \frac{1}{2}\left\|\hat{\beta}\right\|_2^2, \\ \mu_1(\hat{u}) &:= g\hat{h}^T\hat{q} \quad \text{and} \quad \mu_2(\hat{u}) := \tilde{\mu}_2\left(\hat{\mathcal{Y}}^{-1}(\hat{u})\right) := \frac{1}{2}\hat{\beta}^T\mathcal{P}_1(\hat{\omega})\hat{\beta}. \end{aligned} \tag{3.5}$$

4. Construction of entropy conservative numerical fluxes.

This is the main part of this current work. We will construct symmetric, consistent, entropy-conservative numerical two-point fluxes according to [50, 51]. Therefore, we need to remind the flux function (3.4) to compute the flux potential. We denote again by v the entropy variable and μ is the entropy flux. Then, the flux potential is defined through

$$\psi(v) := \langle v, f(v)\rangle - \mu(u(v)).$$

To construct the numerical flux, we have to express the above expression in terms according to the used variables. Therefore, we need the entropy variable $\mathrm{D}_{\hat{u}}\eta(\hat{u})$ expressed in conservative variables. We recall Definition 4 and get

$$\mathrm{D}_{\hat{\mathsf{v}}}\eta(\hat{\mathsf{v}}) = \left(g\hat{h}^T - \frac{1}{2}\hat{\mathsf{v}}^{2^T}, \hat{\mathsf{v}}^T\right) := (v_1^T, v_2^T), \quad \text{with } v_1, v_2 \in \mathbb{R}^{K+1}, \tag{4.1}$$

which leads us to

$$\hat{h} = \frac{v_1 + \frac{1}{2}v_2^2}{g}, \;\; \hat{h} \in \mathbb{R}^{K+1}. \tag{4.2}$$

Note that v_1, v_2 are vectors and by a slide abuse of notation we denote v_2^2 to express as well a vector. Then, the sum is well-defined. This definition helps us to treat the vector-valued expression similar to the deterministic case in a component-wise sense. For further computations, we need to put the following property down on v_2^2.

Lemma 6. *It holds for the in Definition 4 introduced velocity $v_2 = \hat{\mathsf{v}} = \mathcal{P}^{-1}(\hat{\alpha})\hat{\beta}$ with its particularly defined square $v_2^2 = \hat{\mathsf{v}}^2 = \mathcal{P}_2(\hat{\omega})\hat{\beta}$ for every $\gamma \in \mathbb{R}^{k+1}$: $(v_2^2)^T\mathcal{P}(\gamma)v_2 = v_2^T\mathcal{P}(\gamma)v_2^2$.*

Proof. The poof is simply calling the definitions and applying property three from the $\mathcal{A}$gPC bases 1. We get

$$(v_2^2)^T\mathcal{P}(\gamma)v_2 = \hat{\beta}^T\mathcal{P}^{-2}(\hat{\alpha})\mathcal{P}(\hat{\beta})\mathcal{P}(\gamma)\mathcal{P}^{-1}(\hat{\alpha}\hat{\beta}) = \hat{\beta}\mathcal{P}^{-1}(\hat{\alpha})\mathcal{P}(\gamma)\mathcal{P}(\hat{\beta})\mathcal{P}^{-2}(\hat{\alpha})\hat{\beta} = v_2^T\mathcal{P}(\gamma)v_2^2. \qquad \blacksquare$$

We insert now this formulation of $\hat{h}$ (4.2) into the flux function. So we have the flux function in terms of the entropy variables. This allows us to go on with the construction. So, we receive

$$\hat{f}_1\left(\hat{u}(v)\right) = \begin{pmatrix} g\mathcal{P}\left(\frac{v_1+\frac{1}{2}v_2^2}{g}\right)v_2 \\ \frac{1}{2}g\left(\frac{v_1+\frac{1}{2}v_2^2}{g} * \frac{v_1+\frac{1}{2}v_2^2}{g}\right) \end{pmatrix} = \frac{1}{g}\begin{pmatrix} \mathcal{P}\left(v_1 + \frac{1}{2}v_2^2\right)v_2 \\ \frac{1}{2}\mathcal{P}\left(v_1 + \frac{1}{2}v_2^2\right)\left(v_1 + \frac{1}{2}v_2^2\right) \end{pmatrix}$$

and

$$\hat{f}_2\left(\hat{u}(v)\right) = \begin{pmatrix} 0 \\ \frac{v_1+\frac{1}{2}v_2^2}{g} * v \end{pmatrix} = \frac{1}{g}\begin{pmatrix} 0 \\ \mathcal{P}\left(v_1 + \frac{1}{2}v_2^2\right)v_2^2 \end{pmatrix}.$$

We apply the variable v from (4.1) on our entropy flux μ (3.5) and by reformulation, we get

$$\mu(\hat{u}) = g\hat{h}^T\hat{q} + \frac{1}{2}\hat{\beta}^T\mathcal{P}_1(\hat{\omega})\hat{\beta} \overset{*}{=} g\hat{h}^T\mathcal{P}(\hat{h})v_2 + \frac{1}{2}\hat{\beta}^T\mathcal{P}(\hat{\beta})\mathcal{P}^{-1}(\hat{\alpha})\hat{\beta}$$
$$= g\hat{h}^T\mathcal{P}(\hat{h})v_2 + \frac{1}{2}(\hat{\beta}*\hat{\beta})v_2 = g\hat{h}^T\mathcal{P}(\hat{h})v_2 + \frac{1}{2}\left(v_2^2\right)^T\mathcal{P}(\hat{h})v_2.$$

For $*$, we apply for the first term that $\hat{\mathsf{v}}^T(\hat{\omega})\mathcal{P}(\hat{h}) = (\hat{\alpha}*\hat{\beta})^T$ holds and for the second one, we use just the definition of $\mathcal{P}_1(\hat{\omega})$. The next line follows from Definition 4. The last follows from this short computation

$$\hat{\beta}*\hat{\beta} = \mathcal{P}^2(\hat{\alpha})\mathcal{P}^{-2}(\hat{\alpha})\mathcal{P}(\hat{\beta})\hat{\beta} = \mathcal{P}(\hat{h})\mathcal{P}(\hat{\beta})\mathcal{P}^{-2}(\hat{\alpha})\hat{\beta} = \mathcal{P}(\hat{h})v_2^2 = \left(v_2^2\right)^T\mathcal{P}(\hat{h}).$$

So we apply the expression of $\hat{h}$ in entropy variables (4.2), and receive the entropy flux μ in entropy variables. It is given through

$$\mu\left(\hat{u}(v)\right) = \left(v_1^T + \frac{1}{2}v_2^{2^T}\right)\mathcal{P}\left(\frac{v_1+\frac{1}{2}v_2^2}{g}\right)v_2 + \frac{1}{2g}\left(v_2^2\right)^T\mathcal{P}\left(v_1 + \frac{1}{2}v_2^2\right)v_2.$$

195 For further simplification, we use that the argument appears linear in every entry of $\mathcal{P}$

$$\mathcal{P}\left(v_1 + \frac{1}{2}v_2^2\right)v_2 = \left(\mathcal{P}(v_1) + \frac{1}{2}\mathcal{P}(v_2^2)\right)v_2 = \mathcal{P}(v_1)v_2 + \frac{1}{2}\mathcal{P}(v_2^2)v_2.$$

Following this scheme, we get

$$\hat{f}\left(\hat{u}(v)\right) = \frac{1}{g}\begin{pmatrix}\mathcal{P}(v_1)v_2 + \frac{1}{2}\mathcal{P}(v_2^2)v_2\\ \frac{1}{2}\mathcal{P}(v_1)v_1 + \frac{3}{2}\mathcal{P}(v_1)v_2^2 + \frac{5}{8}\mathcal{P}(v_2^2)v_2^2\end{pmatrix}.$$

4.1. Flux potential. Due to our considerations before, we are now able to construct an explicit formula for the flux potential in component-wise formulations. This leads to the foundation of our flux construction later. We first derive a general formula for the flux potential. From the condition $\psi = \mathbf{v^T\hat{f}} - \mu$, we obtain by simple calculations

$$\begin{aligned}\psi &= \frac{1}{g}\mathbf{v}^T\begin{pmatrix}\mathcal{P}(v_1)v_2 + \frac{1}{2}\mathcal{P}(v_2^2)v_2\\ \frac{1}{2}\mathcal{P}(v_1)v_1 + \frac{3}{2}\mathcal{P}(v_1)v_2^2 + \frac{5}{8}\mathcal{P}(v_2^2)v_2^2\end{pmatrix}\\
&\quad - \frac{1}{g}\left(v_1^T\mathcal{P}(v_1)v_2 + \frac{1}{2}(v_2^2)^T\mathcal{P}(v_1)v_2 + v_1^T\frac{1}{2}\mathcal{P}(v_2^2)v_2 + \frac{1}{4}(v_2^2)^T\mathcal{P}(v_2^2)v_2\right)\\
&\quad - \frac{1}{2g}\left(\left(v_2^2\right)^T\mathcal{P}(v_1)v_2 + \frac{1}{2}\left(v_2^2\right)^T\mathcal{P}(v_2^2)v_2\right)\\
&= \frac{1}{g}\Bigg(v_1^T\mathcal{P}(v_1)v_2 + \frac{1}{2}v_1^T\mathcal{P}(v_2^2)v_2 + \frac{1}{2}v_2^T\mathcal{P}(v_1)v_1 + \frac{3}{2}v_2^T\mathcal{P}(v_1)v_2^2 + \frac{5}{8}v_2^T\mathcal{P}(v_2^2)v_2^2\\
&\quad - \Bigg(v_1^T\mathcal{P}(v_1)v_2 + \frac{1}{2}(v_2^2)^T\mathcal{P}(v_1)v_2 + \frac{1}{2}v_1^T\mathcal{P}(v_2^2)v_2\\
&\quad + \frac{1}{4}(v_2^2)^T\mathcal{P}(v_2^2)v_2 + \frac{1}{2}(v_2^2)^T\mathcal{P}(v_1)v_2 + \frac{1}{4}(v_2^2)^T\mathcal{P}(v_2^2)v_2\Bigg)\Bigg)\end{aligned}$$

$$= \frac{1}{2g}\left(v_2^T\mathcal{P}(v_1)v_1 + v_2^T\mathcal{P}(v_1)v_2^2 + \frac{1}{4}v_2^T\mathcal{P}(v_2^2)v_2^2\right) = \frac{1}{2g}\left(v_2^T(v_1 * v_1) + v_2^T(v_1 * v_2^2) + \frac{1}{4}v_2^T(v_2^2 * v_2^2)\right).$$

Now we can formulate the following theorem which gives us an explicit expression of the flux potential.

Theorem 7. *The potential ψ for different dimensions K can be explicitly written as*

$$\psi = \frac{1}{2g}\left(\sum_{i,j,k=0}^{K} v_{2_i}v_{1_j}v_{1_k}\langle\phi_i\phi_j\phi_k\rangle + \sum_{i,j,k=0}^{K} v_{2_i}v_{1_j}v_{2_k}^2\langle\phi_i\phi_j\phi_k\rangle + \frac{1}{4}\sum_{i,j,k=0}^{K} v_{2_i}v_{2_j}^2v_{2_k}^2\langle\phi_i\phi_j\phi_k\rangle\right). \tag{4.3}$$

Proof. We remind the matrix formulation of the Galerkin product (2.8). Then we consider formulation (4.1), take the first summand and apply basic matrix-vector or rather a vector vector multiplication

$$v_2^T\mathcal{P}(v_1)v_1 = (v_{2_1},\ldots,v_{2_K})\left(\sum_{k=0}^{K} v_{1_k}M_k^K\right)\begin{pmatrix} v_{1_1}\\ \vdots \\ v_{1_K}\end{pmatrix} = \sum_{k,i,j=0}^{K} v_{2_i}v_{1_k}v_{1_j}M_{k_{i_j}} = \sum_{i,j,k=0}^{K} v_{2_i}v_{1_j}v_{1_k}\langle\phi_i\phi_j\phi_k\rangle.$$

Analogous calculations for the remaining summands lead us to the desired result. ■

Having the flux potential, we can now follow [37] to obtain an entropy-conservative numerical flux. We denote with $\overline{v} := \frac{v(+)+v(-)}{2}$ the mean value and $[[v]] := v(+) - v(-)$ the jump between two states. To receive an entropy conservative numerical flux, it needs to fulfil the equality in (2.3), i.e.

$$[[v]] \cdot f^{\text{num}} = [[\psi]].$$

Therefore, we need to identify $[[\psi]]$, which is not unique here. To derive a formulation, we use the discrete analogue of the product rule $[[v_iv_j]] = \overline{v_i}[[v_j]] + [[v_i]]\overline{v_j}$. Applying our choice of averaging to ψ, we obtain

(4.4)

$$\begin{aligned}
[[\psi]] &= \frac{1}{2g}\left([[v_2^T(v_1 * v_1)]] + [[v_2^T(v_1 * v_2^2)]] + \frac{1}{4}[[v_2^T(v_2^2 * v_2^2)]]\right)\\
&= \frac{1}{2g}\left(\sum_{i,j,k=0}^{K}[[v_{2_i}v_{1_j}v_{1_k}]]\langle\phi_i\phi_j\phi_k\rangle + \sum_{i,j,k=0}^{K}[[v_{2_i}v_{1_j}v_{2_k}^2]]\langle\phi_i\phi_j\phi_k\rangle + \frac{1}{4}\sum_{i,j,k=0}^{K}[[v_{2_i}v_{2_j}^2v_{2_k}^2]]\langle\phi_i\phi_j\phi_k\rangle\right)\\
&= \frac{1}{2g}\left(\sum_{i,j,k=0}^{K}\left([[v_{2_i}v_{1_j}v_{1_k}]] + [[v_{2_i}v_{1_j}v_{2_k}^2]] + \frac{1}{4}[[v_{2_i}v_{2_j}^2v_{2_k}^2]]\right)\langle\phi_i\phi_j\phi_k\rangle\right)\\
&= \frac{1}{2g}\left(\sum_{i,j,k=0}^{K}\left(\overline{v_{2_i}}\;\overline{v_{1_k}}[[v_{1_j}]] + \overline{v_{2_i}}\;\overline{v_{1_j}}[[v_{1_k}]] + [[v_{2_i}]]\overline{v_{1_i}v_{1_k}} + [[v_{1_j}]]\overline{v_{2_i}v_{2_k}^2} + \overline{v_{1_j}}[[v_{2_i}]]\overline{v_{2_k}^2}\right.\right.\\
&\qquad\left.\left. + 2\overline{v_{1_j}}\;\overline{v_{2_i}}[[v_{2_k}]]\overline{v_{2_k}} + \frac{1}{4}[[v_{2_i}]]\overline{v_{2_j}^2v_{2_k}^2} + \frac{1}{2}\overline{v_{2_i}}\;\overline{v_{2_j}}\overline{v_{2_k}^2}[[v_{2_j}]] + \frac{1}{2}\overline{v_{2_i}}\;\overline{v_{2_k}}\overline{v_{2_j}^2}[[v_{2_k}]]\right)\langle\phi_i\phi_j\phi_k\rangle\right)\\
&= \frac{1}{2g}\left(\sum_{i,j,k=0}^{K}\left([[v_{1_j}]](\overline{v_{2_i}}\;\overline{v_{1_k}} + \overline{v_{2_i}v_{2_k}^2}) + [[v_{1_k}]]\overline{v_{1_j}}\;\overline{v_{2_i}} + [[v_{2_i}]](\overline{v_{1_j}v_{1_k}} + \overline{v_{1_j}}\;\overline{v_{2_k}^2} + \frac{1}{4}\overline{v_{2_j}^2v_{2_k}^2})\right.\right.\\
&\qquad\left.\left. + [[v_{2_k}]](2\overline{v_{1_j}}\;\overline{v_{2_i}}\;\overline{v_{2_k}} + \frac{1}{2}\overline{v_{2_i}}\;\overline{v_{2_k}}\overline{v_{2_j}^2}) + [[v_{2_j}]]\frac{1}{2}\overline{v_{2_i}}\;\overline{v_{2_j}}\overline{v_{2_k}^2}\right)\langle\phi_i\phi_j\phi_k\rangle\right).
\end{aligned}$$

Remark 4.1. In Appendix A.1 an explicit representation for $K \in \{0,1,2\}$ determined with a straightforward calculation compared to the results obtained by use of Theorem 4.3 and expression (4.4) is given. Here, we like to point out that for $K=0$, we end up with the purely determined flux potential as described and used in [43]. Therefore, our calculation is obviously in accordance with the purely deterministic case.

4.2. Numerical fluxes. Finally, we can construct entropy conservative numerical fluxes from condition (2.3). We reformulation (2.3) to $[[v_1, v_2]] \cdot \begin{pmatrix} f_1^{\text{num}} \\ f_2^{\text{num}} \end{pmatrix} = [[\psi]]$. and search for the two parts.

There are many possibilities to choose such a numerical flux. We require the condition in a way where we split $[[\psi]]$ in summands depending on $[[v_1]]$ and $[[v_2]]$. We get for the part depending on $[[v_1]]$

$$
\begin{aligned}
[[\psi_1]] &= \frac{1}{2g}\left(\sum_{i,j,k=0}^{K}\left([[v_{1_j}]](\overline{v_{2_i}}\;\overline{v_{1_k}} + \overline{v_{2_i}v_{2_k}^2}) + [[v_{1_k}]]\overline{v_{1_j}}\;\overline{v_{2_i}}\right)\langle\phi_i\phi_j\phi_k\rangle\right. \\
&= \frac{1}{2g}\left(\sum_{j=0}^{K}[[v_{1_j}]]\sum_{i,k=0}^{K}\overline{v_{2_i}}\;\overline{v_{1_k}} + \overline{v_{2_j}v_{2_k}^2} + \sum_{k=0}^{K}[[v_{1_k}]]\sum_{i,j=0}^{K}\overline{v_{1_j}}\;\overline{v_{2_i}}\right)\langle\phi_i\phi_j\phi_k\rangle.
\end{aligned}
\tag{4.5}
$$

An indices transformation leads us to

$$
[[\psi_1]] = \frac{1}{2g}\sum_{n=0}^{K}[[v_{1_n}]]\sum_{i,j=0}^{K}\left(\overline{v_{1_j}}\;\overline{v_{2_i}} + \overline{v_{2_i}}\;\overline{v_{1_j}} + \overline{v_{2_i}v_{2_0}^2}\right)\langle\phi_i\phi_j\phi_{i_n}\rangle.
$$

Additionally to that, we have vectors $v_1, v_2 \in \mathbb{R}^{K+1}$, so we get through the definition of the scalar product the sum $[[v_1]] \cdot f_1^{\text{num}} = \sum_{n=0}^{K} v_{1_n} f_{1_n}^{\text{num}}$. For simplicity, we choose the approach where we consider every single $f_{1_n}^{\text{num}}$. Thus, we have the first component of our flux

$$
\begin{aligned}
[[v_{1_n}]]f_{1_n}^{\text{num}} &= [[v_{1_n}]]\frac{1}{2g}\sum_{i,j=0}^{K}\left(\overline{v_{1_j}}\;\overline{v_{2_i}} + \overline{v_{2_i}}\;\overline{v_{1_j}} + \overline{v_{2_i}v_{2_j}^2}\right)\langle\phi_i\phi_j\phi_n\rangle \\
\Leftrightarrow f_{1_n}^{\text{num}} &= \frac{1}{2g}\sum_{i,j=0}^{K}\left(\overline{v_{1_j}}\;\overline{v_{2_i}} + \overline{v_{2_i}}\;\overline{v_{1_j}} + \overline{v_{2_i}v_{2_j}^2}\right)\langle\phi_i\phi_j\phi_n\rangle.
\end{aligned}
$$

For the second part including $[[v_2]]$, we have

$$
\begin{aligned}
[[\psi_2]] = \frac{1}{2g}\sum_{i,j,k=0}^{K}\Bigg(&[[v_{2_i}]](\overline{v_{1_j}v_{1_k}} + \overline{v_{1_j}}\;\overline{v_{2_k}^2} + \frac{1}{4}\overline{v_{2_j}^2v_{2_k}^2}) + [[v_{2_k}]](2\overline{v_{1_j}}\;\overline{v_{2_i}}\;\overline{v_{2_k}} \\
&+ \frac{1}{2}\overline{v_{2_i}}\;\overline{v_{2_k}}\overline{v_{2_j}^2}) + [[v_{2_j}]]\frac{1}{2}\overline{v_{2_i}}\;\overline{v_{2_j}}\overline{v_{2_k}^2}\Bigg)\langle\phi_i\phi_j\phi_k\rangle,
\end{aligned}
$$

which leads to variable transformation to

$$
\begin{aligned}
[[v_{2_n}]]f_{2_n}^{\text{num}} &= [[v_{2_n}]]\frac{1}{2g}\sum_{i,j=0}^{K}\left(\overline{v_{1_j}v_{1_i}} + \overline{v_{1_j}}\;\overline{v_{2_i}^2} + 2\overline{v_{1_j}}\;\overline{v_{2_i}}\;\overline{v_{2_n}} + \frac{1}{4}\overline{v_{2_j}^2v_{2_i}^2} + \overline{v_{2_i}}\;\overline{v_{2_n}}\overline{v_{2_j}^2}\right)\langle\phi_i\phi_j\phi_n\rangle \\
\Leftrightarrow f_{2_n}^{\text{num}} &= \frac{1}{2g}\sum_{i,j=0}^{K}\left(\overline{v_{1_j}v_{1_i}} + \overline{v_{1_j}}\;\overline{v_{2_i}^2} + 2\overline{v_{1_j}}\;\overline{v_{2_i}}\;\overline{v_{2_n}} + \frac{1}{4}\overline{v_{2_j}^2v_{2_i}^2} + \overline{v_{2_i}}\;\overline{v_{2_n}}\overline{v_{2_j}^2}\right)\langle\phi_i\phi_j\phi_n\rangle.
\end{aligned}
$$

With this, we have finally constructed our entropy-conservative numerical flux and can formulate the following theorem.

Theorem 8. *Under the above considerations, a numerical entropy conservative flux for the SG-SW system is given in close formula for each component via* $f^{\text{num}} = \left(f^{\text{num}}_{1_0}, \cdots, f^{\text{num}}_{1_K}, f^{\text{num}}_{2_0}, \cdots, f^{\text{num}}_{2_K}\right)^T$ *with*

$$\begin{pmatrix} f^{\text{num}}_{1_n} \\ f^{\text{num}}_{2_n} \end{pmatrix} = \frac{1}{2g}\sum_{i,j=0}^{n} \begin{pmatrix} 2\overline{v_{1_j}}\;\overline{v_{2_i}} + \overline{v_{2_i}v_{2_j}^2} \\ \overline{v_{1_j}v_{1_i}} + \overline{v_{1_j}}\;\overline{v_{2_i}^2} + \frac{1}{4}\overline{v_{2_j}^2 v_{2_i}^2} + 2\overline{v_{1_j}}\;\overline{v_{2_i}}\;\overline{v_{2_n}} + \overline{v_{2_i}}\;\overline{v_{2_n}v_{2_j}^2} \end{pmatrix} \langle \phi_i\phi_j\phi_n\rangle.$$

To get finally an explicit expression, we insert the mean value in in (8) and get for f_1^{num}:

$$\begin{aligned} 2g f^{\text{num}}_{1_{i,j,n}} &= \left(2\overline{v_{1_j}}\;\overline{v_{2_i}} + \overline{v_{2_i}v_{2_j}^2}\right)\langle\phi_i\phi_j\phi_n\rangle \\ &= \left(2\frac{v_{1_{j_-}} + v_{1_{j_+}}}{2}\frac{v_{2_{i_-}} + v_{2_{i_+}}}{2} + \frac{v_{2_{i_-}}v_{2_{j_-}}^2 + v_{2_{i_+}}v_{2_{j_+}}^2}{2}\right)\langle\phi_i\phi_j\phi_n\rangle \\ &= \frac{v_{1_{j_-}}v_{2_{i_-}} + v_{1_{j_-}}v_{2_{i_+}} + v_{1_{j_+}}v_{2_{i_-}} + v_{1_{j_+}}v_{2_{i_+}} + v_{2_{i_-}}v_{2_{j_-}}^2 + v_{2_{i_+}}v_{2_{j_+}}^2}{2}\langle\phi_i\phi_j\phi_n\rangle, \end{aligned}$$

where we obtain for the second part

$$\begin{aligned} 2g f^{\text{num}}_{2_{i,j,n}} &= \left(\overline{v_{1_j}v_{1_i}} + \overline{v_{1_j}}\,\overline{v_{2_i}^2} + \frac{1}{4}\overline{v_{2_j}^2 v_{2_i}^2} + 2\overline{v_{1_j}}\;\overline{v_{2_i}}\;v_{2_{n_-}} + \overline{v_{2_i}}\;v_{2_{n_-}}\overline{v_{2_j}^2}\right)\langle\phi_i\phi_j\phi_n\rangle \\ &= \Bigg(\frac{v_{1_{j_+}}v_{1_{i_+}} + v_{1_{j_-}}v_{1_{i_-}}}{2} + \frac{v_{1_{j_+}} + v_{1_{j_-}}}{2}\frac{v_{2_{i_+}}^2 + v_{2_{i_-}}^2}{2} + \frac{1}{4}\frac{v_{2_{i_+}}^2 v_{2_{j_+}}^2 + v_{2_{i_-}}^2 v_{2_{j_-}}^2}{2} \\ &\quad + 2\frac{v_{1_{j_+}} + v_{1_{j_-}}}{2}\frac{v_{2_{i_+}} + v_{2_{i_-}}}{2}\frac{v_{2_{n_+}} + v_{2_{n_-}}}{2} + \frac{v_{2_{i_+}} + v_{2_{i_-}}}{2}\frac{v_{2_{n_+}} + v_{2_{n_-}}}{2}\frac{v_{2_{j_+}}^2 + v_{2_{j_-}}^2}{2}\Bigg)\langle\phi_i\phi_j\phi_n\rangle \\ &= \frac{1}{2}\Bigg((v_{1_{j_+}}v_{1_{i_+}} + v_{1_{j_-}}v_{1_{i_-}}) + \frac{1}{2}(v_{1_{j_+}}v_{2_{i_+}}^2 + v_{1_{j_+}}v_{2_{i_-}}^2 + v_{1_{j_-}}v_{2_{i_+}}^2 + v_{1_{j_-}}v_{2_{i_-}}^2) \\ &\quad + \frac{1}{4}(v_{2_{j_+}}^2 v_{2_{i_+}}^2 + v_{2_{j_-}}^2 v_{2_{i_-}}^2) \\ &\quad + \frac{1}{2}(v_{1_{j_+}}v_{2_{i_+}}v_{2_{n_+}} + v_{1_{j_+}}v_{2_{i_-}}v_{2_{n_+}} + v_{1_{j_-}}v_{2_{i_+}}v_{2_{n_+}} + v_{1_{j_-}}v_{2_{i_-}}v_{2_{n_+}} \\ &\quad + v_{1_{j_+}}v_{2_{i_+}}v_{2_{n_-}} + v_{1_{j_+}}v_{2_{i_-}}v_{2_{n_-}} + v_{1_{j_-}}v_{2_{i_+}}v_{2_{n_-}} + v_{1_{j_-}}v_{2_{i_-}}v_{2_{n_-}}) \\ &\quad + \frac{1}{4}(v_{2_{i_+}}v_{2_{n_+}}v_{2_{j_+}}^2 + v_{2_{i_+}}v_{2_{n_-}}v_{2_{j_+}}^2 + v_{2_{i_-}}v_{2_{n_+}}v_{2_{j_+}}^2 + v_{2_{i_-}}v_{2_{n_-}}v_{2_{j_+}}^2 \\ &\quad + v_{2_{i_+}}v_{2_{n_+}}v_{2_{j_-}}^2 + v_{2_{i_+}}v_{2_{n_-}}v_{2_{j_-}}^2 + v_{2_{i_-}}v_{2_{n_+}}v_{2_{j_-}}^2 + v_{2_{i_-}}v_{2_{n_-}}v_{2_{j_-}}^2)\Bigg)\langle\phi_i\phi_j\phi_n\rangle. \end{aligned}$$

Before we apply this flux in our numerical scheme, we give the following example for explanation.

Example 9. *If $K = 0$, we are again in the purely deterministic case. By direct calculations, we obtain*

$$f^{\text{num}} = \begin{pmatrix} f^{\text{num}}_{1_0} \\ f^{\text{num}}_{2_0} \end{pmatrix} = \frac{1}{2g}\begin{pmatrix} 2\overline{v_1}\;\overline{v_2} + \overline{v_2^3} \\ \overline{v_1^2} + \overline{v_1}\,\overline{v_2^2} + \frac{1}{4}\overline{v_2^4} + 2\overline{v_1}\;\overline{v_2}^2 + \overline{v_2}^2\overline{v_2^2} \end{pmatrix}.$$

This is exactly the flux described in [43] with parameters $a_1 = a_2 = 1$. This can be also seen with the explicit representation, where we obtain

$$f_1^{\text{num}} = \frac{1}{2g}\left(v_{1_-}v_{2_-} + v_{1_-}v_{2_+} + v_{1_+}v_{2_-} + v_{1_+}v_{2_+} + v_{2_-}^3 + v_{2_+}^3\right)$$

for the first part and

$$\begin{aligned} f_2^{\text{num}} = \frac{1}{2g}\Bigg(&\frac{1}{2}(v_{1_+}^2 + v_{1_-}^2) + \frac{1}{4}(v_{1_+}v_{2_+}^2 + v_{1_+}v_{2_-}^2 + v_{1_-}v_{2_+}^2 + v_{1_-}v_{2_-}^2) \\ &+ \frac{1}{8}(v_{2_+}^4 + v_{2_-}^4) + \frac{1}{4}(v_{1_+}v_{2_+}^2 + v_{1_+}v_{2_-}v_{2_+} + v_{1_-}v_{2_+}^2 \\ &+ v_{1_-}v_{2_-}v_{2_+} + v_{1_+}v_{2_+}v_{2_-} + v_{1_+}v_{2_-}^2 + v_{1_-}v_{2_+}v_{2_-} + v_{1_-}v_{2_-}^2) \\ &+ (v_{2_+}^4 + v_{2_+}^3 v_{2_-} + v_{2_-}v_{2_+}^3 + v_{2_-}^2 v_{2_+}^2 + v_{2_+}^2 v_{2_-}^2 + v_{2_+}v_{2_-}^3 + v_{2_-}^3 v_{2_+} + v_{2_-}^4) \\ =& \frac{1}{2}(v_{1_+}^2 + v_{1_-}^2) + \frac{1}{2}(v_{1_+}v_{2_+}^2 + v_{1_+}v_{2_-}^2 + v_{1_-}v_{2_+}^2 + v_{1_-}v_{2_-}^2 + v_{1_+}v_{2_-}v_{2_+} + v_{1_-}v_{2_-}v_{2_+}) \\ &+ \frac{1}{4}(v_{2_+}^4 + v_{2_-}^4 + v_{2_+}^3 v_{2_-} + v_{2_-}^3 v_{2_+} + v_{2_-}^2 v_{2_+} 2)\Bigg) \end{aligned}$$

for the second one. Again this is the same one as in [43] with the selection of $a_1 = a_2 = 1$. By selecting another averaging procedure, we would have delivered another parameter form. Our study is therefore consistent with the purely deterministic case. For $K \in \{1, 2\}$, explicit formulas can be found in the appendix A.2.

In the following section, we have finally constructed entropy-conservative two-point fluxes for our SG-SW system. The idea is based on the development of an entropy flux potential and using condition (2.3). However, we have not included bottom topography in our investigation up to this point. A possible way including it is described in Appendix C.

5. Numerical results.

For the numerical experiments, we use *Julia* [5] with the numerical simulation framework *Trixi* [44, 47]. Several entropy conservative/dissipative high-order schemes can be found in Trixi.jl, in particular the DGSEM with flux differencing introduced in Section 2.1. We apply for the time discretization always the strong-stability preserving Runge-Kutta methods *SSPRK33* method [11]. In this work, we focus on purely academic test cases as a proof of concept, where we demonstrate the entropy conservation, well-balancing for lake-at-rest and the high-order accuracy of our approach. For the Haar type expansion, we select the Haar wavelets as our basis for the probability space. Therefore, we assume a uniform distribution of the random variable. Due to a new close formula for Haar wavelets [26], we can avoid solving a minimization problem in every step which has normally been done to obtain the Galerkin square root [40]. This also increases the efficiency of our approach. We shortly repeat the definition of the Haar wavelets and some of their main properties.

Definition 10. *(Haar Wavelets [40]). The wavelet system is defined by*

$$\psi(\xi) := \begin{cases} 1 & \textit{if} \;\; 0 \le \xi \le 1/2, \\ -1 & \textit{if} \;\; 1/2 \le \xi < 1, \\ 0 & \textit{else}, \end{cases} \tag{5.1}$$

with

$$\psi_{j,k}(\xi) = 2^{j/2}\psi(2^j\xi - k), \quad j = 0, 1, \ldots; \quad k = 0. \ldots, 2^{j-1},$$

This yields through a lexicographical order to basis functions

$$\phi_1 = \psi, \quad \phi_2 = \psi_{1,0} \text{ and } \phi_3 = \psi_{1,1}.$$

We can develop according to [26] a closed formulation for every gPC mode based on the orthogonal eigenvalue decomposition $\mathcal{P}_J(\hat{u}) = \mathcal{H}_J \mathrm{D}_J(\hat{u}) \mathcal{H}_J^T$ with the *classical Haar matrices*

$$\mathcal{H}_J = \begin{bmatrix} \mathcal{H}_{J-1} \otimes (1,1) \\ \mathbf{1} \otimes (1,-1) \end{bmatrix} \quad \text{for} \quad \mathcal{H}_0 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}. \tag{5.2}$$

Theorem 11. *The gPC modes $\hat{\alpha}, \hat{\beta}$ and $\hat{\mathsf{v}}$ are for a gPC expansion with Haar wavelets, given through the closed formulas*

$$\hat{\alpha} = \mathcal{H}\mathrm{D}(\hat{h})^{\frac{1}{2}}\mathcal{H}^T \hat{e}_1, \quad \hat{\beta} = \mathcal{H}\mathrm{D}(\hat{h})^{-\frac{1}{2}}\mathrm{D}(\hat{q})\mathcal{H}^T \hat{e}_1, \quad \hat{\mathsf{v}} = \mathcal{H}\mathrm{D}(\hat{h})^{-1}\mathrm{D}(\hat{q})\mathcal{H}^T \hat{e}_1,$$

with the classical Haar matrices (5.2)*.*

The proof can be found in Appendix B for completeness and was given first up-to-our knowledge in [26]. We are now able to construct numerical test cases. Again, we focus in this paper on purely basic simulations and consider some generic test cases to give a first proof of concept. More realistic experiments, a comparison and numerical analysis will be done in future works. We give here only results for selecting $K = 3$ which denotes the four term series expansion. For simplification, we consider always in our test the gravitational constant $g = 1$. We fix a few further parameters which will not be changed in our experiments. The mesh on our simulation domain $[-1, 1]$ is built with an initial refinement of 5, which means the original mesh, consisting of 2 cells, is 2^5 times refined, and we obtain a grid of $2^{5+1} = 64$ cells. We choose for our solver the DGSEM method with polynomials of degree 2 and our entropy-conservative numerical flux as surface flux and also as volume flux. Therefore, we test here only for entropy conservation. The initial conditions change for every experiments, but we use periodic boundary conditions always. We test if our numerical fluxes preserve the *lake at rest* state, study convergence and entropy-conserving properties on the model of a simple wave. Consider first the *lake at rest* state. For numerical methods of SW equations, it is important to preserve this steady state which is referred as well as well-balancing property[1]. It means, that in case of a flat, not moving water surface no velocity will occur over time.

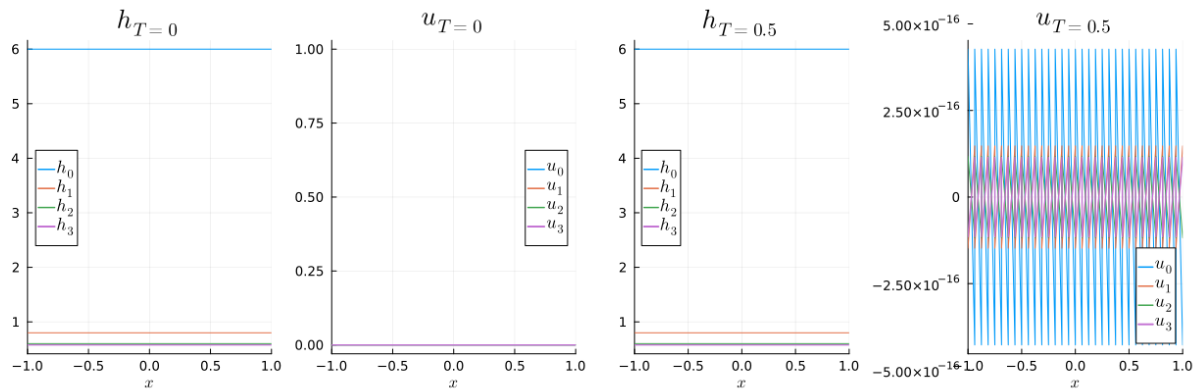
Experiment 11.1. *Lake at rest. We consider the initial conditions*

$$h_0 = 6, \quad h_1 = 0.8, \quad h_2 = 0.6, \quad h_3 = 0.577, \quad q_0 = q_1 = q_2 = q_3 = 0,$$

without any velocity, observed until $T = 0.5$ with time step $\Delta t = 10^{-3}$. The initial conditions are chosen according to the uniform distribution $\mathcal{U}(4, 8)$ following [24]. So h_0 describes the expected value and the sum over the squares of the higher coefficients, the variance (2.2)*.*

We observe in Figure 1 that neither in the velocity (up to errors in machine precision $\approx 10^{-16}$) nor the water heights any movements can be recognized during our simulations. So, Experiment 11.1 confirms our result. The here developed method preserves the steady state and is is well-balanced. In the next test, we consider some variations in water height and momentum.

[1] We refer to [32] and references therein for more information about well-balancing and capturing further equilibria.



(a) $T = 0$ (b) $T = 0.5$

Figure 1: Experiment 11.1

Experiment 11.2. *We start with a flat water surface and model a sinus wave on it, represented trough the following initial conditions*

$$h_0 = 6, \quad h_1 = 0.8, \quad h_2 = 0.6, \quad h_3 = 0.577, \quad q_0 = q_1 = q_2 = q_3 = \frac{1}{2}\sin(\pi x).$$

Again, the initial conditions are derived from a uniform distribution $\mathcal{U}(4, 8)$ following [24]. The water starts moving over time, which can be observed in Figure 2 but no discontinuous are built.

We take a look at the entropy. We aimed for an entropy conservative method which means we expect no change in entropy over time. We integrate the time derivative of the entropy over the domain Ω in every time step and give the plot in Figure 3. We recognize that the change of entropy is in machine precision. This verifies numerically that our method is indeed entropy conservative.

Finally, in our last test, we want to investigate the high-order accuracy of our DGSEM approach in space. Here, we simulate a sinus wave on an already nonflat water surface

Experiment 11.3.

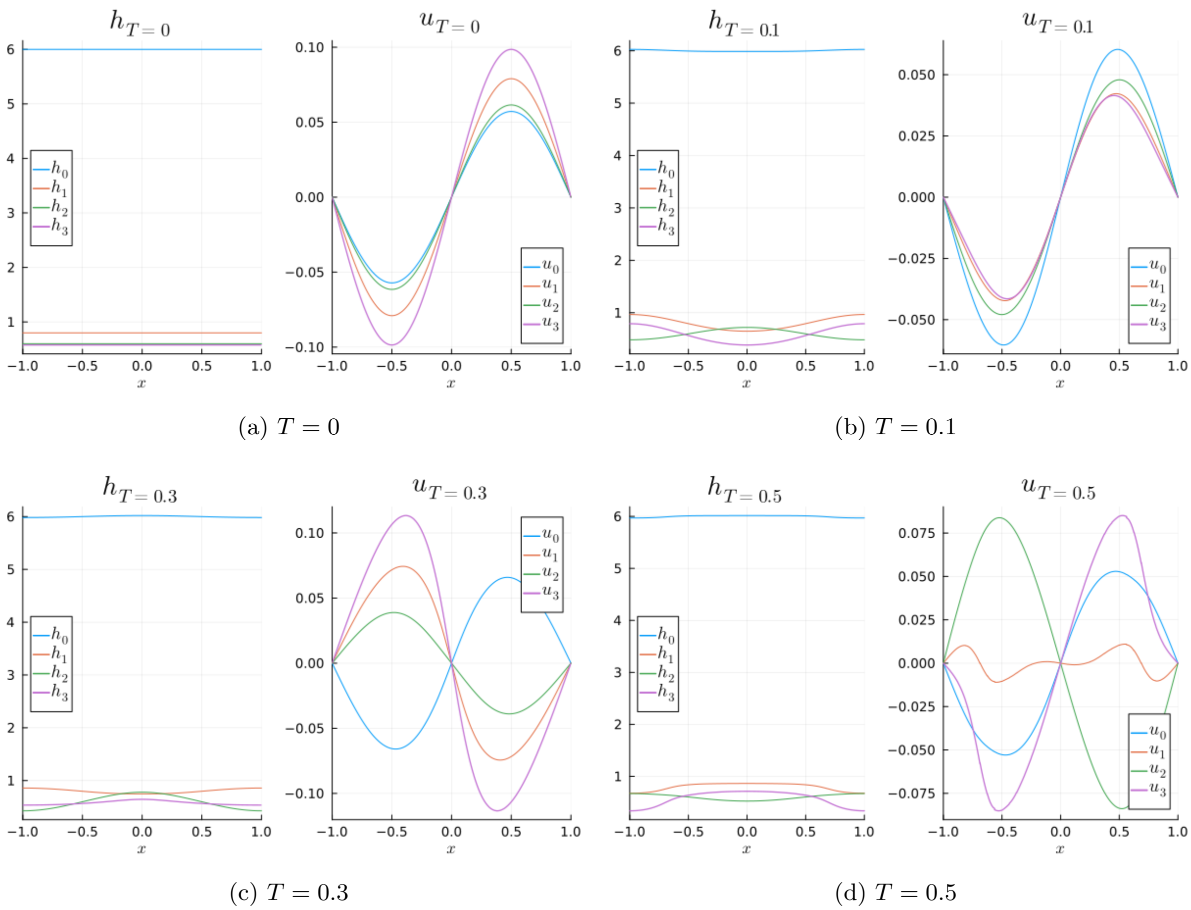
$$h_0 = 6 - q_0, \quad h_1 = 0.8 - q_1, \quad h_2 = 0.6 - q_2, \quad h_3 - q_3 = 0.577, \quad q_0 = q_1 = q_2 = q_3 = \sin(\frac{\pi x}{2})^2,$$

which shows figure 4

We consider the errors, referring to the solution of finest mesh h_{ref}, in L^2 and L^∞ norm on our computational domain $\Omega = [-1, 1]$, where N denotes the original number of grid cells and $2N$ the next higher refinement.

The convergence analysis runs for initial conditions from experiment 11.3 with additional quantities time step size $\Delta t = 10^{-9}$ and final time $T = 10^{-8}$. So, we observe 100 time steps. The initial mesh refinement is 2, which means the mesh contains 8 cells, and we go over 4 iterations/refinements. The time step is chosen this small to guarantee the error of the solver of the time-dependent ODE will not mask the error of our DGSEM method for the spatial discretization.

The convergence table 5 confirms a third-order experimental order of convergence (EOC). This behaviour has been expected and also verified. Similar results can be observed while changing the polynomial degree in the solver which gives us always the desired order.



(a) $T = 0$ (b) $T = 0.1$

(c) $T = 0.3$ (d) $T = 0.5$

Figure 2: Experiment 11.2

6. Summary and outlook. In this paper, we have developed an entropy-conservative and well-balanced DG method for the shallow water equations with uncertainties. We used the SG approach in the context of UQ and by applying the Roe variable transformation, we could re-write our stochastic system in a purely deterministic hyperbolic one. Due to recent investigations by Herty and Gerster [24], an entropy flux pair is known for this augmented system and we derived from this pair a corresponding entropy flux potential in closed form. We use this entropy flux potential in the following to construct for the first time entropy conservative numerical fluxes for the SG-SW system. By applying these fluxes in the DGSEM with flux differencing, we finally obtain an entropy conservative scheme. In our first academic numerical experiments, we have considered lake-at-rest, convergence in space and entropy behaviours. Our scheme is well-balanced for lake-at-rest, high-order accurate in space and entropy conservative which was of particular interest. Up to this point, we have considered only academic test cases to give proof of concept. In future work, we will extend our investigation. First, we will consider also nonzero bottom topography and test our scheme for well-balancing and entropy conservation. The next aspect which is already under investigation is the consideration of more advanced tests including discontinuities like in stochastic damn-break. Here, we use the entropy dissipative formulation by combining our split scheme with entropy dissipative numerical surface fluxes (local Lax-Friedrich, etc.). However, also the extension of positivity-preserving limiting strategies will be important if more

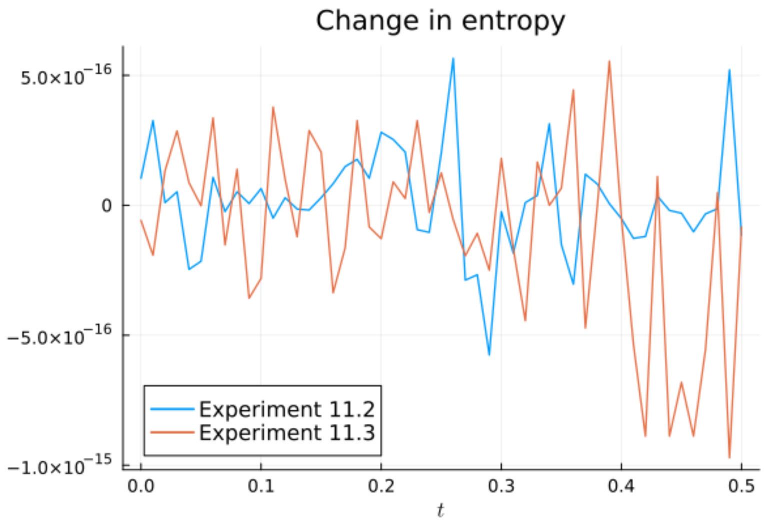
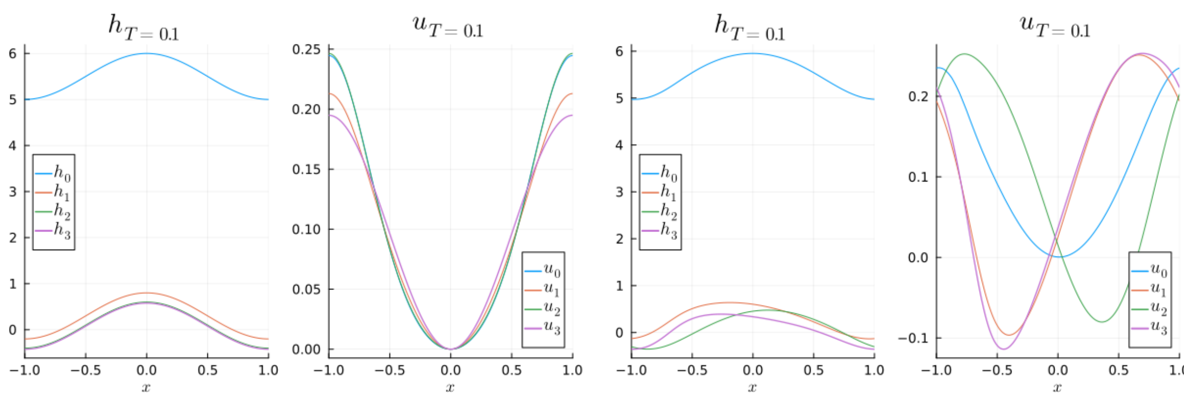


Figure 3: Change in entropy in Experiment 11.2 and Experiment 11.3.



(a) $T = 0$ (b) $T = 0.1$

Figure 4: Experiment 11.3

advanced/realistic experiments are considered. Additionally, the convergence analysis concerning the Galerkin expansion has to be studied and comparison analysis to other techniques [28, 29, 41, 57, 61] will be as well considered in future work. Especially the non-intrusive approach from [61] is of particular interest. In our intrusive method, all stochastic integrals are calculated exactly before the calculations where in [61] a pseudo-spectral ansatz is used with suitable quadrature rules in the stochastic space where the underlying deterministic solver in this paper and the one in [61] is the same.
In this manuscript, we have only considered the well-balancing property with respect to the lake-at-rest state. More generalized approaches to ensure other equilibria are also of interest and are already investigated in the context of SW with uncertainty in [12, 13].

Acknowledgements. The authors like to thank Stephan Gerster for some discussion about entropy-entropy flux pairs and Haar wavelet expension. P.Ö. also gratefully acknowledge support of the Gutenberg Research College, JGU Mainz.

N	h_0		h_1		h_2		h_3	
	error	EOC	error	EOC	error	EOC	error	EOC
8	$7.60 \cdot 10^{-3}$	–	$7.60 \cdot 10^{-3}$	–	$7.60 \cdot 10^{-3}$	–	$7.60 \cdot 10^{-3}$	–
16	$9.76 \cdot 10^{-4}$	2.96	$9.76 \cdot 10^{-4}$	2.96	$9.76 \cdot 10^{-4}$	2.96	$9.76 \cdot 10^{-4}$	2.96
32	$1.23 \cdot 10^{-4}$	2.99	$1.23 \cdot 10^{-4}$	2.99	$1.23 \cdot 10^{-4}$	2.99	$1.23 \cdot 10^{-4}$	2.99
64	$1.54 \cdot 10^{-5}$	3.00	$1.54 \cdot 10^{-5}$	3.00	$1.54 \cdot 10^{-5}$	3.00	$1.54 \cdot 10^{-5}$	3.00

(a) EOC table of h in L^2-norm

N	h_0		h_1		h_2		h_3	
	error	EOC	error	EOC	error	EOC	error	EOC
8	$1.15 \cdot 10^{-2}$	–	$1.15 \cdot 10^{-2}$	–	$1.15 \cdot 10^{-2}$	–	$1.15 \cdot 10^{-2}$	–
16	$1.77 \cdot 10^{-3}$	2.71	$1.77 \cdot 10^{-3}$	2.70	$1.77 \cdot 10^{-3}$	2.70	$1.77 \cdot 10^{-3}$	2.71
32	$2.32 \cdot 10^{-4}$	2.93	$2.32 \cdot 10^{-4}$	2.93	$2.32 \cdot 10^{-4}$	2.93	$2.32 \cdot 10^{-4}$	2.93
64	$2.94 \cdot 10^{-5}$	2.98	$2.95 \cdot 10^{-5}$	2.98	$2.96 \cdot 10^{-5}$	2.98	$2.95 \cdot 10^{-5}$	2.98

(b) EOC table of h in L^∞-norm

N	q_0		q_1		q_2		q_3	
	error	EOC	error	EOC	error	EOC	error	EOC
8	$7.60 \cdot 10^{-3}$	–	$7.60 \cdot 10^{-3}$	–	$7.60 \cdot 10^{-3}$	–	$7.60 \cdot 10^{-3}$	–
16	$9.76 \cdot 10^{-4}$	2.96	$9.76 \cdot 10^{-4}$	2.96	$9.76 \cdot 10^{-4}$	2.96	$9.76 \cdot 10^{-4}$	2.96
32	$1.23 \cdot 10^{-4}$	2.99	$1.23 \cdot 10^{-4}$	2.99	$1.23 \cdot 10^{-4}$	2.99	$1.23 \cdot 10^{-4}$	2.99
64	$1.54 \cdot 10^{-5}$	3.00	$1.54 \cdot 10^{-5}$	3.00	$1.54 \cdot 10^{-5}$	3.00	$1.54 \cdot 10^{-5}$	3.00

(c) EOC table of q in L^2-norm

N	q_0		q_1		q_2		q_3	
	error	EOC	error	EOC	error	EOC	error	EOC
8	$1.15 \cdot 10^{-2}$	–	$1.16 \cdot 10^{-2}$	–	$1.16 \cdot 10^{-2}$	–	$1.16 \cdot 10^{-2}$	–
16	$1.77 \cdot 10^{-3}$	2.71	$1.77 \cdot 10^{-3}$	2.70	$1.77 \cdot 10^{-3}$	2.70	$1.77 \cdot 10^{-3}$	2.70
32	$2.32 \cdot 10^{-4}$	2.93	$2.33 \cdot 10^{-4}$	2.93	$2.33 \cdot 10^{-4}$	2.93	$2.33 \cdot 10^{-4}$	2.93
64	$2.94 \cdot 10^{-5}$	2.98	$3.03 \cdot 10^{-5}$	2.94	$3.02 \cdot 10^{-5}$	2.95	$3.03 \cdot 10^{-5}$	2.94

(d) EOC table of q in L^∞-norm

Figure 5: EOC tables for the conservative variables water height h and momentum q

Appendix A. Explicit representations.

In the following, we give for completeness and a better comprehension some examples for the first expansions of the flux potential and numerical flux functions. In particular, the numerical flux functions are explicitly constructed. On that way, we understand that the formula from Theorem 7 indeed delivers the same results as obtained through straight forward calculations.

A.1. Flux potential.

We will give in the following an explicit representation of the flux potential (4.4) for $K \in \{0, 1, 2\}$ determined with a straight forward calculation compared to the formula from Theorem 7. Then we do the averaging for these results and compare them to the one obtained by use of (4.4).
In the deterministic case we have no more a vector v but a scalar. We need it as a control case to prove if our calculations

are consistent with existing results. It leads us to

$$\hat{f}\left(\hat{u}(v)\right) = \frac{1}{g}\begin{pmatrix} v_1 v_2 + \frac{1}{2}v_2^3 \\ \frac{1}{2}v_1^2 + \frac{3}{2}v_1 v_2^2 + \frac{5}{8}v_2^4, \end{pmatrix}$$

for the flux and

$$\psi = \frac{1}{2}v_2\left(v_1^2 + v_1 v_2^2 + \frac{1}{4}v_2^4\right),$$

which are the same as in [43].
We remind for $K=1$ the original formulation

$$[[\psi]] = \frac{1}{2g}\left([[v_2^T(v_1 * v_1)]] + [[v_2^T(v_1 * v_2^2)]] + \frac{1}{4}[[v_2^T(v_2^2 * v_2^2)]]\right),$$

and begin with the straight forward calculations. We precompute for the first summand of $[[\psi]]$

$$\begin{aligned} v_2^T(v_1 * v_1) &= v_2^T\begin{pmatrix} v_{1_0}^2\langle\phi_0^3\rangle + 2v_{1_0}v_{1_1}\langle\phi_0^2\phi_1\rangle + v_{1_1}^2\langle\phi_0\phi_1^2\rangle \\ v_{1_0}^2\langle\phi_0^2\phi_1\rangle + 2v_{1_0}v_{1_1}\langle\phi_1^2\phi_0\rangle + v_{1_1}^2\langle\phi_1^3\rangle \end{pmatrix} \\ &= v_{2_0}v_{1_0}^2\langle\phi_0^3\rangle + 2v_{2_0}v_{1_0}v_{1_1}\langle\phi_0^2\phi_1\rangle + v_{2_0}v_{1_1}^2\langle\phi_0\phi_1^2\rangle + v_{2_1}v_{1_0}^2\langle\phi_0^2\phi_1\rangle \\ &\quad + 2v_{2_1}v_{1_0}v_{1_1}\langle\phi_1^2\phi_0\rangle + v_{2_1}v_{1_1}^2\langle\phi_1^3\rangle, \end{aligned}$$

which is indeed the same as formula (4.4) delivers. The next step is to carry out the averaging

$$\begin{aligned} [[v_2^T(v_1 * v_1)]] = [[v_{2_0}v_{1_0}^2]]\langle\phi_0^3\rangle + 2[[v_{2_0}v_{1_0}v_{1_1}]]\langle\phi_0^2\phi_1\rangle + [[v_{2_0}v_{1_1}^2]]\langle\phi_0\phi_1^2\rangle \\ + [[v_{2_1}v_{1_0}^2]]\langle\phi_0^2\phi_1\rangle + 2[[v_{2_1}v_{1_0}v_{1_1}]]\langle\phi_1^2\phi_0\rangle + [[v_{2_1}v_{1_1}^2]]\langle\phi_1^3\rangle. \end{aligned}$$

For a greater clarity, we will do it for the different parts

$$\begin{aligned} [[v_{2_0}v_{1_0}^2]] &= \overline{v_{2_0}}[[v_{1_0}^2]] + [[v_{2_0}]]\overline{v_{1_0}^2} = 2\overline{v_{2_0}}[[v_{1_0}]]\overline{v_{1_0}} + [[v_{2_0}]]\overline{v_{1_0}^2} \\ [[v_{2_0}v_{1_0}v_{1_1}]] &= \overline{v_{2_0}}[[v_{1_0}v_{1_1}]] + [[v_{2_0}]]\overline{v_{1_0}v_{1_1}} = \overline{v_{2_0}}[[v_{1_0}]]\overline{v_{1_1}} + \overline{v_{2_0}}[[v_{1_1}]]\overline{v_{1_0}} + [[v_{2_0}]]\overline{v_{1_0}v_{1_1}} \\ [[v_{2_0}v_{1_1}^2]] &= 2\overline{v_{2_0}}[[v_{1_1}]]\overline{v_{1_1}} + [[v_{2_0}]]\overline{v_{1_1}^2} \\ [[v_{2_1}v_{1_0}^2]] &= 2\overline{v_{2_1}}[[v_{1_0}]]\overline{v_{1_0}} + [[v_{2_1}]]\overline{v_{1_0}^2} \\ [[v_{2_1}v_{1_0}v_{1_1}]] &= \overline{v_{2_1}}[[v_{1_0}]]\overline{v_{1_1}} + \overline{v_{2_1}}[[v_{1_1}]]\overline{v_{1_0}} + [[v_{2_1}]]\overline{v_{1_0}v_{1_1}} \\ [[v_{2_1}v_{1_1}^2]] &= 2\overline{v_{2_1}}[[v_{1_1}]]\overline{v_{1_1}} + [[v_{2_1}]]\overline{v_{1_1}^2}. \end{aligned}$$

For the remaining summands we proceed similar to the first one

$$\begin{aligned} v_2^T(v_1 * v_2^2) &= v_2^T\begin{pmatrix} v_{1_0}v_{2_0}^2\langle\phi_0^3\rangle + v_{1_0}v_{2_1}^2\langle\phi_0^2\phi_1\rangle + v_{1_1}v_{2_0}^2\langle\phi_0^2\phi_1\rangle + v_{1_1}v_{2_1}^2\langle\phi_1^2\phi_0\rangle \\ v_{1_0}v_{2_0}^2\langle\phi_0^2\phi_1\rangle + v_{1_0}v_{2_1}^2\langle\phi_1^2\phi_0\rangle + v_{1_1}v_{2_0}^2\langle\phi_0\phi_1^2\rangle + v_{1_1}v_{2_1}^2\langle\phi_1^3\rangle \end{pmatrix} \\ &= v_{1_0}v_{2_0}^3\langle\phi_0^3\rangle + v_{1_0}v_{2_0}v_{2_1}^2\langle\phi_0^2\phi_1\rangle + v_{1_1}v_{2_0}^3\langle\phi_0^2\phi_1\rangle + v_{1_1}v_{2_0}v_{2_1}^2\langle\phi_1^2\phi_0\rangle \\ &\quad + v_{1_0}v_{2_0}^2v_{2_1}\langle\phi_0^2\phi_1\rangle + v_{1_0}v_{2_1}^3\langle\phi_1^2\phi_0\rangle + v_{1_1}v_{2_0}^2v_{2_1}\langle\phi_0\phi_1^2\rangle + v_{1_1}v_{2_1}^3\langle\phi_1^3\rangle, \end{aligned}$$

with averaging as follows

$$\begin{aligned}
[[v_2^T(v_1 * v_2^2)]] &= [[v_{1_0}v_{2_0}^3]]\langle\phi_0^3\rangle + [[v_{1_0}v_{2_0}v_{2_1}^2]]\langle\phi_0^2\phi_1\rangle + [[v_{1_1}v_{2_0}^3]]\langle\phi_0^2\phi_1\rangle \\
&\quad + [[v_{1_1}v_{2_0}v_{2_1}^2]]\langle\phi_1^2\phi_0\rangle + [[v_{1_0}v_{2_0}^2v_{2_1}]]\langle\phi_0^2\phi_1\rangle \\
&\quad + [[v_{1_0}v_{2_1}^3]]\langle\phi_1^2\phi_0\rangle + [[v_{1_1}v_{2_0}^2v_{2_1}]]\langle\phi_0\phi_1^2\rangle + [[v_{1_1}v_{2_1}^3]]\langle\phi_1^3\rangle, \\
[[v_{1_0}v_{2_0}^3]] &= [[v_{1_0}]]\overline{v_{2_0}^3} + \overline{v_{1_0}}[[v_{2_0}^3]] \\
&= [[v_{1_0}]]\overline{v_{2_0}^3} + 2\overline{v_{1_0}}\ \overline{v_{2_0}}^2[[v_{2_0}]] + \overline{v_{1_0}}\overline{v_{2_0}^2}[[v_{2_0}]] \\
[[v_{1_0}v_{2_0}v_{2_1}^2]] &= [[v_{1_0}]]\overline{v_{2_0}v_{2_1}^2} + \overline{v_{1_0}}[[v_{2_0}v_{2_1}^2]] \\
&= [[v_{1_0}]]\overline{v_{2_0}v_{2_1}^2} + \overline{v_{1_0}}[[v_{2_0}]]\overline{v_{2_1}^2} + 2\overline{v_{1_0}}\ \overline{v_{2_0}}[[v_{2_1}]]\overline{v_{2_1}} \\
[[v_{1_1}v_{2_0}^3]] &= [[v_{1_1}]]\overline{v_{2_0}^3} + 2\overline{v_{1_1}}\ \overline{v_{2_0}}^2[[v_{2_0}]] + \overline{v_{1_1}}\overline{v_{2_0}^2}[[v_{2_0}]] \\
[[v_{1_1}v_{2_0}v_{2_1}^2]] &= [[v_{1_1}]]\overline{v_{2_0}v_{2_1}^2} + \overline{v_{1_1}}[[v_{2_0}]]\overline{v_{2_1}^2} + 2\overline{v_{1_1}}\ \overline{v_{2_0}}[[v_{2_1}]]\overline{v_{2_1}} \\
[[v_{1_0}v_{2_0}^2v_{2_1}]] &= [[v_{1_0}]]\overline{v_{2_1}v_{2_0}^2} + \overline{v_{1_0}}[[v_{2_1}]]\overline{v_{2_0}^2} + 2\overline{v_{1_0}}\ \overline{v_{2_1}}[[v_{2_0}]]\overline{v_{2_0}} \\
[[v_{1_0}v_{2_1}^3]] &= [[v_{1_0}]]\overline{v_{2_1}^3} + 2\overline{v_{1_0}}\ \overline{v_{2_1}}^2[[v_{2_1}]] + \overline{v_{1_0}}\overline{v_{2_1}^2}[[v_{2_1}]] \\
[[v_{1_1}v_{2_0}^2v_{2_1}]] &= [[v_{1_1}]]\overline{v_{2_1}v_{2_0}^2} + \overline{v_{1_1}}[[v_{2_1}]]\overline{v_{2_0}^2} + 2\overline{v_{1_1}}\ \overline{v_{2_1}}[[v_{2_0}]]\overline{v_{2_0}} \\
[[v_{1_1}v_{2_1}^3]] &= [[v_{1_1}]]\overline{v_{2_1}^3} + 2\overline{v_{1_1}}\ \overline{v_{2_1}}^2[[v_{2_1}]] + \overline{v_{1_1}}\overline{v_{2_1}^2}[[v_{2_1}]].
\end{aligned}$$

It lasts the part

$$\begin{aligned}
v_2^T(v_2^2 * v_2^2) &= v_2^T\begin{pmatrix} v_{2_0}^4\langle\phi_0^3\rangle + 2v_{2_0}^2v_{2_1}^2\langle\phi_0^2\phi_1\rangle + v_{2_1}^4\langle\phi_1^2\phi_0\rangle \\ v_{2_0}^4\langle\phi_0^2\phi_1\rangle + 2v_{2_0}^2v_{2_1}^2\langle\phi_1^2\phi_0\rangle + v_{2_1}^4\langle\phi_1^3\rangle \end{pmatrix} \\
&= v_{2_0}^5\langle\phi_0^3\rangle + 2v_{2_0}^3v_{2_1}^2\langle\phi_0^2\phi_1\rangle + v_{2_0}v_{2_1}^4\langle\phi_1^2\phi_0\rangle \\
&\quad + v_{2_1}v_{2_0}^4\langle\phi_0^2\phi_1\rangle + 2v_{2_0}^2v_{2_1}^3\langle\phi_1^2\phi_0\rangle + v_{2_1}^5\langle\phi_1^3\rangle,
\end{aligned}$$

with an averaging of

$$\begin{aligned}
[[v_2^T(v_2^2 * v_2^2)]] &= [[v_{2_0}^5]]\langle\phi_0^3\rangle + 2[[v_{2_0}^3v_{2_1}^2]]\langle\phi_0^2\phi_1\rangle + [[v_{2_0}v_{2_1}^4]]\langle\phi_1^2\phi_0\rangle \\
&\quad + [[v_{2_1}v_{2_0}^4]]\langle\phi_0^2\phi_1\rangle + 2[[v_{2_0}^2v_{2_1}^3]]\langle\phi_1^2\phi_0\rangle + [[v_{2_1}^5]]\langle\phi_1^3\rangle,
\end{aligned}$$

given through

$$\begin{aligned}
[[v_{2_0}^5]] &= [[v_{2_0}]]\overline{v_{2_0}^4} + 2\overline{v_{2_0}}\overline{v_{2_0}^2}[[v_{2_0}^2]] = [[v_{2_0}]]\overline{v_{2_0}^4} + 4\overline{v_{2_0}}^2\overline{v_{2_0}^2}[[v_{2_0}]] \\
[[v_{2_0}^3v_{2_1}^2]] &= [[v_{2_0}]]\overline{v_{2_0}^2v_{2_1}^2} + [[v_{2_0}^2v_{2_1}^2]]\overline{v_{2_0}} \\
&= [[v_{2_0}]]\overline{v_{2_0}^2v_{2_1}^2} + 2[[v_{2_0}]]\overline{v_{2_1}^2}\ \overline{v_{2_0}}^2 + 2[[v_{2_1}]]\overline{v_{2_0}^2}\overline{v_{2_0}}\ \overline{v_{2_1}} \\
[[v_{2_0}v_{2_1}^4]] &= [[v_{2_0}]]\overline{v_{2_1}^4} + \ 2\overline{v_{2_0}}[[v_{2_1}^2]]\overline{v_{2_1}^2} = [[v_{2_0}]]\overline{v_{2_1}^4} + 4\overline{v_{2_0}}[[v_{2_1}]]\overline{v_{2_1}^2}\overline{v_{2_1}} \\
[[v_{2_1}v_{2_0}^4]] &= [[v_{2_1}]]\overline{v_{2_0}^4} + 4\overline{v_{2_1}}[[v_{2_0}]]\overline{v_{2_0}^2}\overline{v_{2_0}} \\
[[v_{2_1}^3v_{2_0}^2]] &= [[v_{2_1}]]\overline{v_{2_1}^2v_{2_0}^2} + 2[[v_{2_1}]]\overline{v_{2_0}^2}\ \overline{v_{2_1}}^2 + 2[[v_{2_0}]]\overline{v_{2_1}^2}\overline{v_{2_1}}\ \overline{v_{2_0}} \\
[[v_{2_1}^5]] &= [[v_{2_1}]]\overline{v_{2_1}^4} + 4\overline{v_{2_1}}^2\overline{v_{2_1}^2}[[v_{2_1}]].
\end{aligned}$$

Finally we have

$$
\begin{aligned}
[[\psi]]_1 = \frac{1}{2g}\Bigg(&\langle\phi_0^3\rangle\Bigg([[v_{1_0}]]2\overline{v_{2_0}}\ \overline{v_{1_0}} + [[v_{2_0}]]\overline{v_{1_0}^2} + [[v_{1_0}]]\overline{v_{2_0}^3} + 2\overline{v_{1_0}}\ \overline{v_{2_0}}^2[[v_{2_0}]] + \overline{v_{1_0}}\overline{v_{2_0}^2}[[v_{2_0}]] \\
&+ \frac{1}{4}[[v_{2_0}]]\overline{v_{2_0}^4} + \overline{v_{2_0}}^2\overline{v_{2_0}^2}[[v_{2_0}]]\Bigg) \\
&+ \langle\phi_0^2\phi_1\rangle\Bigg(2\big(\overline{v_{2_0}}[[v_{1_0}]]\overline{v_{1_1}} + \overline{v_{2_0}}[[v_{1_1}]]\overline{v_{1_0}} + [[v_{2_0}]]\overline{v_{1_0}v_{1_1}}\big) + 2\overline{v_{2_1}}[[v_{1_0}]]\overline{v_{1_0}} + [[v_{2_1}]]\overline{v_{1_0}^2} \\
&+ [[v_{1_0}]]\overline{v_{2_0}v_{2_1}^2} + \overline{v_{1_0}}[[v_{2_0}]]\overline{v_{2_1}^2} + 2\overline{v_{1_0}}\ \overline{v_{2_0}}[[v_{2_1}]]\overline{v_{2_1}} \\
&+ [[v_{1_1}]]\overline{v_{2_0}^3} + 2\overline{v_{1_1}}\ \overline{v_{2_0}}^2[[v_{2_0}]] + \overline{v_{1_1}}\overline{v_{2_0}^2}[[v_{2_0}]] \\
&+ [[v_{1_0}]]\overline{v_{2_1}v_{2_0}^2} + \overline{v_{1_0}}[[v_{2_1}]]\overline{v_{2_0}^2} + 2\overline{v_{1_0}}\ \overline{v_{2_1}}[[v_{2_0}]]\overline{v_{2_0}} \\
&+ \frac{1}{2}[[v_{2_0}]]\overline{v_{2_0}^2v_{2_1}^2} + [[v_{2_0}]]\overline{v_{2_1}^2}\overline{v_{2_0}}^2 + [[v_{2_1}]]\overline{v_{2_0}^2}\overline{v_{2_0}}\ \overline{v_{2_1}} \\
&+ \frac{1}{4}[[v_{2_1}]]\overline{v_{2_0}^4} + \overline{v_{2_1}}[[v_{2_0}]]\overline{v_{2_0}^2}\overline{v_{2_0}}\Bigg) \\
&+ \langle\phi_0\phi_1^2\rangle\Bigg(2\overline{v_{2_0}}[[v_{1_1}]]\overline{v_{1_1}} + [[v_{2_0}]]\overline{v_{1_1}^2} + 2\big(\overline{v_{2_1}}[[v_{1_0}]]\overline{v_{1_1}} + \overline{v_{2_1}}[[v_{1_1}]]\overline{v_{1_0}} \\
&+ [[v_{2_1}]]\overline{v_{1_0}v_{1_1}}\big) + [[v_{1_1}]]\overline{v_{2_0}v_{2_1}^2} + \overline{v_{1_1}}[[v_{2_0}]]\overline{v_{2_1}^2} + 2\overline{v_{1_1}v_{2_0}}[[v_{2_1}]]\overline{v_{2_1}} \\
&+ [[v_{1_0}]]\overline{v_{2_1}^3} + 2\overline{v_{1_0}}\ \overline{v_{2_1}}^2[[v_{2_1}]] + \overline{v_{1_0}}\overline{v_{2_1}^2}[[v_{2_1}]] \\
&+ [[v_{1_1}]]\overline{v_{2_1}v_{2_0}^2} + \overline{v_{1_1}}[[v_{2_1}]]\overline{v_{2_0}^2} + 2\overline{v_{1_1}}\ \overline{v_{2_1}}[[v_{2_0}]]\overline{v_{2_0}} \\
&+ \frac{1}{4}[[v_{2_0}]]\overline{v_{2_1}^4} + \overline{v_{2_0}}[[v_{2_1}]]\overline{v_{2_1}^2}\overline{v_{2_1}} + \frac{1}{2}[[v_{2_1}]]\overline{v_{2_1}^2v_{2_0}^2} + [[v_{2_1}]]\overline{v_{2_0}^2}\overline{v_{2_1}}^2 + [[v_{2_0}]]\overline{v_{2_1}^2}\overline{v_{2_1}}\ \overline{v_{2_0}}\Bigg) \\
&+ \langle\phi_1^3\rangle\Bigg(2\overline{v_{2_1}}[[v_{1_1}]]\overline{v_{1_1}} + [[v_{2_1}]]\overline{v_{1_1}^2} \\
&+ [[v_{1_1}]]\overline{v_{2_1}^3} + 2\overline{v_{1_1}}\ \overline{v_{2_1}}^2[[v_{2_1}]] + \overline{v_{1_1}}\overline{v_{2_1}^2}[[v_{2_1}]] + \frac{1}{4}[[v_{2_1}]]\overline{v_{2_1}^4} + \overline{v_{2_1}}^2\overline{v_{2_1}^2}[[v_{2_1}]]\Bigg)\Bigg),
\end{aligned}
$$

which is the same as we get through formula (4.4).

We do the same straight forward calculations for $K=2$ and compute

$$
\begin{aligned}
v_2^T(v_1 * v_1) &= v_2^T\mathcal{P}(v_1)v_1\\
&= v_2^T\begin{pmatrix}
v_{1_0}\langle\phi_0^3\rangle + v_{1_1}\langle\phi_0^2\phi_1\rangle + v_{1_2}\langle\phi_0^2\phi_2\rangle & v_{1_0}\langle\phi_0^2\phi_1\rangle + v_{1_1}\langle\phi_0\phi_1^2\rangle + v_{1_2}\langle\phi_0\phi_1\phi_2\rangle & v_{1_0}\langle\phi_0^2\phi_2\rangle + v_{1_1}\langle\phi_0\phi_1\phi_2\rangle + v_{1_2}\langle\phi_0\phi_2^2\rangle\\
v_{1_1}\langle\phi_0\phi_1^2\rangle + v_{1_0}\langle\phi_0^2\phi_1\rangle + v_{1_2}\langle\phi_0\phi_1\phi_2\rangle & v_{1_0}\langle\phi_0\phi_1^2\rangle + v_{1_1}\langle\phi_1^3\rangle + v_{1_2}\langle\phi_1^2\phi_2\rangle & v_{1_0}\langle\phi_0\phi_1\phi_2\rangle + v_{1_1}\langle\phi_1^2\phi_2\rangle + v_{1_2}\langle\phi_1\phi_2^2\rangle\\
v_{1_2}\langle\phi_0\phi_2^2\rangle + v_{1_1}\langle\phi_0\phi_1\phi_2\rangle + v_{1_0}\langle\phi_0^2\phi_2\rangle & v_{1_0}\langle\phi_0\phi_1\phi_2\rangle + v_{1_1}\langle\phi_1^2\phi_2\rangle + v_{1_2}\langle\phi_1\phi_2^2\rangle & v_{1_0}\langle\phi_1\phi_2^2\rangle + v_{1_1}\langle\phi_1\phi_2^2\rangle + v_{1_2}\langle\phi_2^3\rangle
\end{pmatrix}\begin{pmatrix} v_{1_0}\\ v_{1_1}\\ v_{1_2}\end{pmatrix}\\
&= v_2^T\begin{pmatrix}
v_{1_0}^2\langle\phi_0^3\rangle + 2v_{1_0}v_{1_1}\langle\phi_0^2\phi_1\rangle + 2v_{1_0}v_{1_2}\langle\phi_0^2\phi_2\rangle + 2v_{1_1}v_{1_2}\langle\phi_0\phi_1\phi_2\rangle + v_{1_1}^2\langle\phi_0\phi_1^2\rangle + v_{1_2}^2\langle\phi_0\phi_2^2\rangle\\
v_{1_0}^2\langle\phi_0^2\phi_1\rangle + 2v_{1_0}v_{1_1}\langle\phi_0\phi_1^2\rangle + 2v_{1_0}v_{1_2}\langle\phi_0\phi_1\phi_2\rangle + 2v_{1_1}v_{1_2}\langle\phi_1^2\phi_2\rangle + v_{1_1}^2\langle\phi_1^3\rangle + v_{1_2}^2\langle\phi_1\phi_2^2\rangle\\
v_{1_0}^2\langle\phi_0^2\phi_2\rangle + 2v_{1_0}v_{1_2}\langle\phi_0\phi_2^2\rangle + 2v_{1_0}v_{1_1}\langle\phi_0\phi_1\phi_2\rangle + 2v_{1_1}v_{1_2}\langle\phi_1\phi_2^2\rangle + v_{1_1}^2\langle\phi_1^2\phi_2\rangle + v_{1_2}^2\langle\phi_2^3\rangle
\end{pmatrix}\\
&= v_{2_0}v_{1_0}^2\langle\phi_0^3\rangle + 2v_{2_0}v_{1_0}v_{1_1}\langle\phi_0^2\phi_1\rangle + 2v_{2_0}v_{1_0}v_{1_2}\langle\phi_0^2\phi_2\rangle\\
&\quad + 2v_{2_0}v_{1_1}v_{1_2}\langle\phi_0\phi_1\phi_2\rangle + v_{2_0}v_{1_1}^2\langle\phi_0\phi_1^2\rangle + v_{2_0}v_{1_2}^2\langle\phi_0\phi_2^2\rangle + v_{2_1}v_{1_0}^2\langle\phi_0^2\phi_1\rangle\\
&\quad + 2v_{2_1}v_{1_0}v_{1_1}\langle\phi_0\phi_1^2\rangle + 2v_{2_1}v_{1_0}v_{1_2}\langle\phi_0\phi_1\phi_2\rangle + 2v_{2_1}v_{1_1}v_{1_2}\langle\phi_1^2\phi_2\rangle\\
&\quad + v_{2_1}v_{1_1}^2\langle\phi_1^3\rangle + v_{2_1}v_{1_2}^2\langle\phi_1\phi_2^2\rangle + v_{2_2}v_{1_0}^2\langle\phi_0^2\phi_2\rangle + 2v_{2_2}v_{1_0}v_{1_2}\langle\phi_0\phi_2^2\rangle\\
&\quad + 2v_{2_2}v_{1_0}v_{1_1}\langle\phi_0\phi_1\phi_2\rangle + 2v_{2_2}v_{1_1}v_{1_2}\langle\phi_1\phi_2^2\rangle + v_{2_2}v_{1_1}^2\langle\phi_1^2\phi_2\rangle + v_{2_2}v_{1_2}^2\langle\phi_2^3\rangle.
\end{aligned}
$$

There are six terms which are the same as in case $K=1$ completed by twelve additional terms. The averaging could be done analogous to $K=1$.

The next summand leads to

$$
\begin{aligned}
v_2^T(v_1 * v_2^2) &= v_2^T\mathcal{P}(v_1)v_2^2\\
&= v_2^T\begin{pmatrix}
v_{2_0}^2(v_{1_0}\langle\phi_0^3\rangle + v_{1_1}\langle\phi_0^3\phi_1\rangle + v_{1_2}\langle\phi_0^2\phi_2\rangle & +v_{2_1}^2(v_{1_0}\langle\phi_0^2\phi_1\rangle + v_{1_1}\langle\phi_0\phi_1^2\rangle + v_{1_2}\langle\phi_0\phi_1\phi_2\rangle & +v_{2_2}^2(v_{1_0}\langle\phi_0^2\phi_2\rangle + v_{1_1}\langle\phi_0\phi_1\phi_2\rangle + v_{1_2}\langle\phi_0\phi_2^2\rangle)\\
v_{2_0}^2(v_{1_1}\langle\phi_0\phi_1^2\rangle + v_{1_0}\langle\phi_0^2\phi_1\rangle + v_{1_2}\langle\phi_0\phi_1\phi_2\rangle & +v_{2_1}^2(v_{1_0}\langle\phi_0\phi_1^2\rangle + v_{1_1}\langle\phi_1^3\rangle + v_{1_2}\langle\phi_1^2\phi_2\rangle & +v_{2_2}^2(v_{1_0}\langle\phi_0\phi_1\phi_2\rangle + v_{1_1}\langle\phi_1^2\phi_2\rangle + v_{1_2}\langle\phi_1\phi_2^2\rangle)\\
v_{2_0}^2(v_{1_2}\langle\phi_0\phi_2^2\rangle + v_{1_1}\langle\phi_0\phi_1\phi_2\rangle + v_{1_0}\langle\phi_0^2\phi_2\rangle & +v_{2_1}^2(v_{1_0}\langle\phi_0\phi_1\phi_2\rangle + v_{1_1}\langle\phi_1^2\phi_2\rangle + v_{1_2}\langle\phi_1\phi_2^2\rangle & +v_{2_2}^2(v_{1_0}\langle\phi_0\phi_2^2\rangle + v_{1_1}\langle\phi_1\phi_2^2\rangle + v_{1_2}\langle\phi_2^3\rangle)
\end{pmatrix}\\
&= v_{2_0}^3v_{1_0}\langle\phi_0^3\rangle + v_{1_1}v_{2_0}^3\langle\phi_0^2\phi_1\rangle + v_{1_2}v_{2_0}^3\langle\phi_0^2\phi_2\rangle\\
&\quad + v_{2_1}^2v_{2_0}v_{1_0}\langle\phi_0^2\phi_1\rangle + v_{2_1}^2v_{2_0}v_{1_1}\langle\phi_0\phi_1^2\rangle + v_{2_0}v_{2_1}^2v_{1_2}\langle\phi_0\phi_1\phi_2\rangle\\
&\quad + v_{2_2}^2v_{1_0}v_{2_0}\langle\phi_0^2\phi_2\rangle + v_{2_0}v_{2_2}^2v_{1_1}\langle\phi_0\phi_1\phi_2\rangle + v_{2_0}v_{2_2}^2v_{1_2}\langle\phi_0\phi_2^2\rangle\\
&\quad + v_{2_1}v_{2_0}^2v_{1_1}\langle\phi_0\phi_1^2\rangle + v_{2_1}v_{1_0}v_{2_0}^2\langle\phi_0^2\phi_1\rangle + v_{2_1}v_{2_0}^2v_{1_2}\langle\phi_0\phi_1\phi_2\rangle\\
&\quad + v_{2_1}^3v_{1_0}\langle\phi_0\phi_1^2\rangle + v_{2_1}^3v_{1_1}\langle\phi_1^3\rangle + v_{2_1}^3v_{1_2}\langle\phi_1^2\phi_2\rangle\\
&\quad + v_{2_1}v_{2_2}^2v_{1_0}\langle\phi_0\phi_1\phi_2\rangle + v_{2_2}^2v_{1_1}v_{2_1}\langle\phi_1^2\phi_2\rangle + v_{2_1}v_{2_2}^2v_{1_2}\langle\phi_1\phi_2^2\rangle\\
&\quad + v_{2_2}v_{2_0}^2v_{1_2}\langle\phi_0\phi_2^2\rangle + v_{2_0}^2v_{1_1}v_{2_2}\langle\phi_0\phi_1\phi_2\rangle + v_{2_0}^2v_{1_0}v_{2_2}\langle\phi_0^2\phi_2\rangle\\
&\quad + v_{2_2}v_{2_1}^2v_{1_0}\langle\phi_0\phi_1\phi_2\rangle + v_{2_2}v_{2_1}^2v_{1_1}\langle\phi_1^2\phi_2\rangle + v_{2_2}v_{2_1}^2v_{1_2}\langle\phi_1\phi_2^2\rangle\\
&\quad + v_{2_2}^3v_{1_0}\langle\phi_0\phi_2^2\rangle + v_{2_2}^3v_{1_1}\langle\phi_1\phi_2^2\rangle + v_{2_2}^3v_{1_2}\langle\phi_2^3\rangle.
\end{aligned}
$$

We obtain for the last one

$$
\begin{aligned}
v_2^T(v_2^2 * v_2^2) &= v_2^T \mathcal{P}(v_2^2) v_2^2 \\
&= v_2^T \begin{pmatrix}
v_{2_0}^4 \langle \phi_0^3 \rangle + v_{2_1}^2 v_{2_0}^2 \langle \phi_0^2 \phi_1 \rangle + v_{2_2}^2 v_{2_0}^2 \langle \phi_0^2 \phi_2 \rangle \\
v_{2_0}^2 v_{2_1}^2 \langle \phi_0 \phi_1^2 \rangle + v_{2_0}^4 \langle \phi_0^2 \phi_1 \rangle + v_{2_0}^2 v_{2_2}^2 \langle \phi_0 \phi_1 \phi_2 \rangle \\
v_{2_0}^2 v_{2_2}^2 \langle \phi_0 \phi_2^2 \rangle + v_{2_0}^2 v_{2_1}^2 \langle \phi_0 \phi_1 \phi_2 \rangle + v_{2_0}^4 \langle \phi_0^2 \phi_2 \rangle \\
+ v_{2_1}^2 v_{2_0}^2 \langle \phi_0^2 \phi_1 \rangle + v_{2_1}^4 \langle \phi_0 \phi_1^2 \rangle + v_{2_2}^2 v_{2_1}^2 \langle \phi_0 \phi_1 \phi_2 \rangle \\
+ v_{2_1}^2 v_{2_0}^2 \langle \phi_0 \phi_1^2 \rangle + v_{2_1}^4 \langle \phi_1^3 \rangle + v_{2_1}^2 v_{2_2}^2 \langle \phi_1^2 \phi_2 \rangle \\
+ v_{2_1}^2 v_{2_0}^2 \langle \phi_0 \phi_1 \phi_2 \rangle + v_{2_1}^4 \langle \phi_1^2 \phi_2 \rangle + v_{2_1}^2 v_{2_2}^2 \langle \phi_1 \phi_2^2 \rangle \\
+ v_{2_2}^2 v_{2_0}^2 \langle \phi_0^2 \phi_2 \rangle + v_{2_2}^2 v_{2_1}^2 \langle \phi_0 \phi_1 \phi_2 \rangle + v_{2_2}^4 \langle \phi_0 \phi_2^2 \rangle \\
+ v_{2_2}^2 v_{2_0}^2 \langle \phi_0 \phi_1 \phi_2 \rangle + v_{2_2}^2 v_{2_1}^2 \langle \phi_1^2 \phi_2 \rangle + v_{2_2}^4 \langle \phi_1 \phi_2^2 \rangle \\
+ v_{2_2}^2 v_{2_0}^2 \langle \phi_0 \phi_2^2 \rangle + v_{2_2}^2 v_{2_1}^2 \langle \phi_1 \phi_2^2 \rangle + v_{2_2}^4 \langle \phi_2^3 \rangle
\end{pmatrix} \\
&= v_{2_0}^5 \langle \phi_0^3 \rangle + 2 v_{2_1} v_{2_0}^3 \langle \phi_0^2 \phi_1 \rangle + 2 v_{2_2}^2 v_{2_0}^3 \langle \phi_0^2 \phi_2 \rangle \\
&\quad + 2 v_{2_2}^2 v_{2_1}^2 v_{2_0} \langle \phi_0 \phi_1 \phi_2 \rangle + v_{2_1}^4 v_{2_0} \langle \phi_0 \phi_1^2 \rangle + v_{2_2}^4 v_{2_0} \langle \phi_0 \phi_2^2 \rangle \\
&\quad + 2 v_{2_0}^2 v_{2_1}^3 \langle \phi_0 \phi_1^2 \rangle + 2 v_{2_0}^2 v_{2_2}^2 v_{2_1} \langle \phi_0 \phi_1 \phi_2 \rangle + 2 v_{2_1}^3 v_{2_2}^2 \langle \phi_1^2 \phi_2 \rangle \\
&\quad + v_{2_0}^4 v_{2_1} \langle \phi_0^2 \phi_1 \rangle + v_{2_1}^5 \langle \phi_1^3 \rangle + v_{2_2}^4 v_{2_1} \langle \phi_1 \phi_2^2 \rangle \\
&\quad + 2 v_{2_0}^2 v_{2_2}^3 \langle \phi_0 \phi_2^2 \rangle + 2 v_{2_0}^2 v_{2_1}^2 v_{2_2} \langle \phi_0 \phi_1 \phi_2 \rangle + 2 v_{2_1}^2 v_{2_2}^3 \langle \phi_1 \phi_2^2 \rangle \\
&\quad + v_{2_0}^4 v_{2_2} \langle \phi_0^2 \phi_2 \rangle + v_{2_1}^4 v_{2_2} \langle \phi_1^2 \phi_2 \rangle + v_{2_2}^5 \langle \phi_2^3 \rangle.
\end{aligned}
$$

Through averaging we get

$$
\begin{aligned}
[[\psi]]_2 = [[\psi]]_1 + \frac{1}{2g} \langle \phi_0^2 \phi_2 \rangle s_{0_2^2} + \langle \phi_1^2 \phi_2 \rangle s_{1_2^2} + \langle \phi_2^2 \phi_0 \rangle s_{2_0^2} \\
+ \langle \phi_2^2 \phi_1 \rangle s_{2_1^2} + \langle \phi_0 \phi_1 \phi_2 \rangle s_{0 1_2} + \langle \phi_2^3 \rangle s_{2^3},
\end{aligned}
$$

with

$$
\begin{aligned}
s_{0_2^2} &= [[v_{1_0}]](2\overline{v_{2_0}}\ \overline{v_{1_2}} + \overline{v_{2_0} v_{2_2}^2} + 2\overline{v_{2_2}}\ \overline{v_{1_0}} + \overline{v_{2_0}^3}) \\
&\quad + [[v_{1_2}]](2\overline{v_{1_0}}\ \overline{v_{2_0}} + \overline{v_{2_0}^3}) \\
&\quad + [[v_{2_0}]](2\overline{v_{1_0} v_{1_2}} + \overline{v_{1_0}}\ \overline{v_{2_2}^2} + \frac{1}{2}\overline{v_{2_0}^2 v_{2_2}^2} + \overline{v_{2_0}}^2\ \overline{v_{2_2}^2} \\
&\quad + \overline{v_{1_2} v_{1_0}} + 2\overline{v_{1_2}}\ \overline{v_{2_0}^2} + \overline{v_{1_2}}\ \overline{v_{2_0}}^2 + 2\overline{v_{1_0}}\ \overline{v_{2_2}}\ \overline{v_{2_0}} + \overline{v_{2_2}}\ \overline{v_{2_0}}\ \overline{v_{2_0}^2}) \\
&\quad + [[v_{2_2}]](2\overline{v_{1_0}}\ \overline{v_{2_0}}\ \overline{v_{2_2}} + \overline{v_{2_0}}\ \overline{v_{2_2}}\ \overline{v_{2_0}^2} + \overline{v_{1_0}^2} + \overline{v_{1_0}}\ \overline{v_{2_0}^2} + \frac{1}{4}\overline{v_{2_0}^4}),
\end{aligned}
$$

$$
\begin{aligned}
s_{1_2^2} &= [[v_{1_1}]](2\overline{v_{2_1}}\ \overline{v_{1_2}} + \overline{v_{2_1} v_{2_2}^2} + 2\overline{v_{2_2}}\ \overline{v_{1_1}} + \overline{v_{2_1}^3}) \\
&\quad + [[v_{1_2}]](2\overline{v_{1_1}}\ \overline{v_{2_1}} + \overline{v_{2_1}^3}) \\
&\quad + [[v_{2_1}]](2\overline{v_{1_1} v_{1_2}} + \overline{v_{1_1}}\ \overline{v_{2_2}^2} + \frac{1}{2}\overline{v_{2_1}^2 v_{2_2}^2} + \overline{v_{2_1}}^2\ \overline{v_{2_2}^2} \\
&\quad + \overline{v_{1_2} v_{1_1}} + \overline{v_{1_2}}\ \overline{v_{2_1}^2} + 2\overline{v_{1_2}}\ \overline{v_{2_1}}^2 + 2\overline{v_{1_1}}\ \overline{v_{2_2}}\ \overline{v_{2_1}} + \overline{v_{2_2}}\ \overline{v_{2_1}}\ \overline{v_{2_1}^2}) \\
&\quad + [[v_{2_2}]](2\overline{v_{1_1}}\ \overline{v_{2_1}}\ \overline{v_{2_2}} + \overline{v_{2_1}}\ \overline{v_{2_2}}\ \overline{v_{2_1}^2} + \overline{v_{1_1}^2} + \overline{v_{1_1}}\ \overline{v_{2_1}^2} + \frac{1}{4}\overline{v_{2_1}^4}),
\end{aligned}
$$

$$
\begin{aligned}
s_{2_0^2} &= [[v_{1_2}]](2\overline{v_{2_2}}\ \overline{v_{1_0}} + \overline{v_{2_2} v_{2_0}^2} + 2\overline{v_{2_0}}\ \overline{v_{1_2}} + \overline{v_{2_2}^3}) \\
&\quad + [[v_{1_0}]](2\overline{v_{1_2}}\ \overline{v_{2_2}} + \overline{v_{2_2}^3}) \\
&\quad + [[v_{2_2}]](2\overline{v_{1_2} v_{1_0}} + \overline{v_{1_2}}\ \overline{v_{2_0}^2} + \frac{1}{2}\overline{v_{2_2}^2 v_{2_0}^2} + \overline{v_{2_2}}^2\ \overline{v_{2_0}^2} \\
&\quad + \overline{v_{1_0} v_{1_2}} + \overline{v_{1_0}}\ \overline{v_{2_2}^2} + 2\overline{v_{1_0}}\ \overline{v_{2_2}}^2 + 2\overline{v_{1_2}}\ \overline{v_{2_0}}\ \overline{v_{2_2}} + \overline{v_{2_0}}\ \overline{v_{2_2}}\ \overline{v_{2_2}^2}) \\
&\quad + [[v_{2_0}]](2\overline{v_{1_2}}\ \overline{v_{2_2}}\ \overline{v_{2_0}} + \overline{v_{2_2}}\ \overline{v_{2_0}}\ \overline{v_{2_2}^2} + \overline{v_{1_2}^2} + \overline{v_{1_2}}\ \overline{v_{2_2}^2} + \frac{1}{4}\overline{v_{2_2}^4}),
\end{aligned}
$$

$$
\begin{aligned}
s_{2_1^2} = {} & [[v_{1_2}]](2\overline{v_{2_2}}\,\overline{v_{1_1}} + \overline{v_{2_2}v_{2_1}^2} + 2\overline{v_{2_1}}\,\overline{v_{1_2}} + \overline{v_{2_2}^3}) \\
& + [[v_{1_1}]](2\overline{v_{1_2}}\,\overline{v_{2_2}} + \overline{v_{2_2}^3}) \\
& + [[v_{2_2}]](2\overline{v_{1_2}v_{1_1}} + \overline{v_{1_2}}\,\overline{v_{2_1}^2} + \frac{1}{2}\overline{v_{2_2}^2 v_{2_1}^2} + \overline{v_{2_2}}^2\,\overline{v_{2_1}^2} \\
& + \overline{v_{1_1}v_{1_2}} + \overline{v_{1_1}}\,\overline{v_{2_2}^2} + 2\overline{v_{1_1}}\,\overline{v_{2_2}}^2 + 2\overline{v_{1_2}}\,\overline{v_{2_1}}\,\overline{v_{2_2}} + \overline{v_{2_1}}\,\overline{v_{2_2}}\,\overline{v_{2_2}^2}) \\
& + [[v_{2_1}]](2\overline{v_{1_2}}\,\overline{v_{2_2}}\,\overline{v_{2_1}} + \overline{v_{2_2}}\,\overline{v_{2_1}}\,\overline{v_{2_2}^2} + \overline{v_{1_2}^2} + \overline{v_{1_2}}\,\overline{v_{2_2}^2} + \frac{1}{4}\overline{v_{2_2}^4}),
\end{aligned}
$$

$$
\begin{aligned}
s_{0_{1_2}} = {} & [[v_{1_0}]](2\overline{v_{1_2}}\,\overline{v_{2_1}} + 2\overline{v_{2_2}}\,\overline{v_{1_1}} + \overline{v_{2_2}v_{2_1}^2} + \overline{v_{2_1}v_{2_2}^2}) \\
& + [[v_{1_1}]](2\overline{v_{2_0}}\,\overline{v_{1_2}} + \overline{v_{2_0}v_{2_2}^2} + 2\overline{v_{1_0}}\,\overline{v_{2_2}} + \overline{v_{2_2}v_{2_0}^2}) \\
& + [[v_{1_2}]](2\overline{v_{1_1}}\,\overline{v_{2_0}} + 2\overline{v_{2_1}}\,\overline{v_{1_0}} + \overline{v_{2_1}v_{2_0}^2} + \overline{v_{2_0}v_{2_1}^2}) \\
& + [[v_{2_0}]](2\overline{v_{1_1}v_{1_2}} + \overline{v_{1_1}}\,\overline{v_{2_2}^2} + \frac{1}{2}\overline{v_{2_1}^2 v_{2_2}^2} + 2\overline{v_{1_2}}\,\overline{v_{2_1}}\,\overline{v_{2_0}} \\
& + \overline{v_{2_1}}\,\overline{v_{2_0}}\,\overline{v_{2_2}^2} + \overline{v_{2_2}}\,\overline{v_{2_0}}\,\overline{v_{2_1}^2} + \overline{v_{1_2}v_{2_1}^2} \\
& + 2\overline{v_{1_1}}\,\overline{v_{2_2}}\,\overline{v_{2_0}} + \frac{1}{2}\overline{v_{2_2}}\,\overline{v_{2_1}}\,\overline{v_{2_0}^2}) \\
& + [[v_{2_1}]](\overline{v_{2_0}}\,\overline{v_{2_1}}\,\overline{v_{2_2}^2} + 2\overline{v_{1_2}v_{1_0}} + \overline{v_{1_2}}\,\overline{v_{2_0}^2} + \frac{1}{2}\overline{v_{2_2}^2 v_{2_0}^2} \\
& + 2\overline{v_{1_0}}\,\overline{v_{2_2}}\,\overline{v_{2_1}} + \frac{1}{2}\overline{v_{2_2}}\,\overline{v_{2_1}}\,\overline{v_{2_0}^2} + 2\overline{v_{1_2}}\,\overline{v_{2_0}}\,\overline{v_{2_1}} + \overline{v_{1_0}}\,\overline{v_{2_2}^2}) \\
& + [[v_{2_2}]](2\overline{v_{1_1}}\,\overline{v_{2_0}}\,\overline{v_{2_2}} + \overline{v_{2_0}}\,\overline{v_{2_2}}\,\overline{v_{2_1}^2} + \overline{v_{2_1}}\,\overline{v_{2_2}}\,\overline{v_{2_0}^2} \\
& + 2\overline{v_{1_0}v_{1_1}} + \overline{v_{1_0}}\,\overline{v_{2_1}^2} + \frac{1}{2}\overline{v_{2_0}^2 v_{2_1}} + 2\overline{v_{1_0}}\,\overline{v_{2_1}}\,\overline{v_{2_2}} + \overline{v_{1_1}}\,\overline{v_{2_0}^2}),
\end{aligned}
$$

$$
\begin{aligned}
s_{2^3} = {} & [[v_{1_2}]](2\overline{v_{2_2}}\,\overline{v_{1_2}} + \overline{v_{2_2}^3}) + [[v_{2_2}]](\overline{v_{1_2}}^2 + \overline{v_{1_0}}\,\overline{v_{2_2}^2} \\
& + \frac{1}{4}\overline{v_{2_2}^4} + 2\overline{v_{1_2}}\,\overline{v_{2_2}}^2 + \overline{v_{2_2}}^2\,\overline{v_{2_2}^2}).
\end{aligned}
$$

A.2. Numerical flux.

We will give again some examples of an explicit formulation for the numerical flux through Definition (8).

- If we set $K = 1$ we get for f_1^{num}

$$
\begin{aligned}
f_{1_0}^{\text{num}} = {} & \frac{1}{2g}(2\overline{v_{1_0}}\,\overline{v_{2_0}} + \overline{v_{2_0}^3})\langle \phi_0^3 \rangle \\
f_{1_1}^{\text{num}} = {} & \frac{1}{2g}\Big((2\overline{v_{1_0}}\,\overline{v_{2_0}} + \overline{v_{2_0}^3} + 2\overline{v_{1_0}}\,\overline{v_{2_1}} + \overline{v_{2_1}v_{2_0}^2} + 2\overline{v_{1_1}}\,\overline{v_{2_0}} + \overline{v_{2_0}v_{2_1}^2})\langle \phi_0^2\phi_1 \rangle \\
& + (2\overline{v_{1_1}}\,\overline{v_{2_1}} + \overline{v_{2_1}^3} + 2\overline{v_{1_1}}\,\overline{v_{2_0}} + \overline{v_{2_0}v_{2_1}^2} + 2\overline{v_{1_0}}\,\overline{v_{2_1}} + \overline{v_{2_1}v_{2_0}^2})\langle \phi_1^2\phi_0 \rangle + (2\overline{v_{1_1}}\,\overline{v_{2_1}} + \overline{v_{2_1}^3})\langle \phi_1^3 \rangle\Big),
\end{aligned}
$$

and

$$
\begin{aligned}
f_{2_0}^{\text{num}} = {} & \frac{1}{2g}(\overline{v_{1_0}^2} + \overline{v_{1_0}}\overline{v_{1_0}^2} + \frac{1}{4}\overline{v_{2_0}^4} + 2\overline{v_{1_0}}\,\overline{v_{2_0}}^2 + \overline{v_{2_0}}^2\overline{v_{2_0}^2})\langle \phi_0^3 \rangle \\
f_{2_1}^{\text{num}} = {} & \frac{1}{2g}\Big(\overline{v_{1_0}^2} + \overline{v_{1_0}}\,\overline{v_{1_0}^2} + \frac{1}{4}\overline{v_{2_0}^4} + 4\overline{v_{1_0}}\,\overline{v_{2_0}}\,\overline{v_{2_1}} + 2\overline{v_{2_0}}\,\overline{v_{2_1}}\,\overline{v_{2_0}^2} + 2\overline{v_{1_0}v_{1_1}} + \overline{v_{1_0}}\,\overline{v_{2_1}^2} + \frac{1}{2}\overline{v_{2_0}^2 v_{2_1}^2} \\
& + \overline{v_{1_1}}\,\overline{v_{2_0}^2} + 2\overline{v_{1_1}}\,\overline{v_{2_0}}^2 + \overline{v_{2_0}}^2\overline{v_{2_1}^2})\langle \phi_0^2\phi_1 \rangle \\
& + (\overline{v_{1_1}^2} + \overline{v_{1_1}}\,\overline{v_{1_1}^2} + \frac{1}{4}\overline{v_{2_1}^4} + 4\overline{v_{1_1}}\,\overline{v_{2_1}}\,\overline{v_{2_0}} + 2\overline{v_{2_1}}\,\overline{v_{2_0}}\,\overline{v_{2_1}^2} + 2\overline{v_{1_0}v_{1_1}} + \overline{v_{1_1}}\,\overline{v_{2_0}^2} + \frac{1}{2}\overline{v_{2_1}^2 v_{2_0}^2} \\
& + \overline{v_{1_0}}\,\overline{v_{2_1}^2} + 2\overline{v_{1_0}}\,\overline{v_{2_1}}^2 + \overline{v_{2_1}}^2\overline{v_{2_0}^2})\langle \phi_1^2\phi_0 \rangle \\
& + (\overline{v_{1_1}^2} + \overline{v_{1_1}}\,\overline{v_{1_1}^2} + \frac{1}{4}\overline{v_{2_1}^4} + 2\overline{v_{1_1}}\,\overline{v_{2_1}}^2 + \overline{v_{2_1}}^2\overline{v_{2_1}^2})\langle \phi_1^3 \rangle\Big),
\end{aligned}
$$

for the second part of the flux.

- For $K = 2$ we get again all entries of $K = 1$ but the following ones in addition

$$
\begin{aligned}
f^{\text{num}}_{1_2} = \frac{1}{2g}\Bigg(&(2\overline{v_{1_0}}\,\overline{v_{2_0}} + \overline{v_{2_0}^3} + 2\overline{v_{1_0}}\,\overline{v_{2_2}} + \overline{v_{2_2}v_{2_0}^2} + 2\overline{v_{1_2}}\,\overline{v_{2_0}} + \overline{v_{2_0}v_{2_2}^2}\langle\phi_0^2\phi_2\rangle \\
&+ (2\overline{v_{1_1}}\,\overline{v_{2_1}} + \overline{v_{2_1}^3} + 2\overline{v_{1_1}}\,\overline{v_{2_2}} + \overline{v_{2_2}v_{2_1}^2} + 2\overline{v_{1_2}}\,\overline{v_{2_1}} + \overline{v_{2_1}v_{2_2}^2})\langle\phi_1^2\phi_2\rangle \\
&+ (2\overline{v_{1_2}}\,\overline{v_{2_2}} + \overline{v_{2_2}^3} + 2\overline{v_{1_2}}\,\overline{v_{2_0}} + \overline{v_{2_0}v_{2_2}^2} + 2\overline{v_{1_0}}\,\overline{v_{2_2}} + \overline{v_{2_2}v_{2_0}^2})\langle\phi_2^2\phi_0\rangle \\
&+ (2\overline{v_{1_2}}\,\overline{v_{2_2}} + \overline{v_{2_2}^3})\langle\phi_2^3\rangle \\
&+ (2\overline{v_{1_1}}\,\overline{v_{2_0}} + \overline{v_{2_0}v_{2_1}^2} + 2\overline{v_{1_2}}\,\overline{v_{2_1}} + \overline{v_{2_1}v_{2_2}^2} + 2\overline{v_{1_0}}\,\overline{v_{2_2}} + \overline{v_{2_2}v_{2_0}^2} \\
&+ 2\overline{v_{1_1}}\,\overline{v_{2_2}} + \overline{v_{2_2}v_{2_1}^2} + 2\overline{v_{1_2}v_{2_0}} + \overline{v_{2_0}v_{2_2}^2} + 2\overline{v_{1_0}}\,\overline{v_{2_1}} + \overline{v_{2_1}v_{2_0}^2})\langle\phi_0\phi_1\phi_2\rangle\Bigg)
\end{aligned}
$$

and

$$
\begin{aligned}
f^{\text{num}}_{2_2} = \frac{1}{2g}\Bigg(&\big(\overline{v_{1_0}^2} + \overline{v_{1_0}}\,\overline{v_{1_0}^2} + \frac{1}{4}\overline{v_{2_0}^4} + 4\overline{v_{1_0}}\,\overline{v_{2_0}}\,\overline{v_{2_2}} + 2\overline{v_{2_0}}\,\overline{v_{2_2}}\,\overline{v_{2_0}^2} + 2\overline{v_{1_0}v_{1_2}} + \overline{v_{1_0}}\,\overline{v_{2_2}^2} \\
&+ \frac{1}{2}\overline{v_{2_0}^2v_{2_2}^2} + \overline{v_{1_2}}\,\overline{v_{2_0}^2} + 2\overline{v_{1_2}}\,\overline{v_{2_0}}^2 + \overline{v_{2_0}}^2\overline{v_{2_0}^2}\big)\langle\phi_0^2\phi_2\rangle \\
&+ \big(\overline{v_{1_1}^2} + \overline{v_{1_1}}\,\overline{v_{1_1}^2} + \frac{1}{4}\overline{v_{2_1}^4} + 4\overline{v_{1_1}}\,\overline{v_{2_1}}\,\overline{v_{2_2}} + 2\overline{v_{2_1}}\,\overline{v_{2_2}}\,\overline{v_{2_1}^2} + 2\overline{v_{1_2}v_{1_1}} + \overline{v_{1_1}}\,\overline{v_{2_2}^2} + \frac{1}{2}\overline{v_{2_1}^2v_{2_2}^2} \\
&+ \overline{v_{1_2}}\,\overline{v_{2_1}^2} + 2\overline{v_{1_2}}\,\overline{v_{2_1}}^2 + \overline{v_{2_1}}^2\overline{v_{2_1}^2}\big)\langle\phi_1^2\phi_2\rangle \\
&+ \big(\overline{v_{1_2}^2} + \overline{v_{1_2}}\,\overline{v_{1_2}^2} + \frac{1}{4}\overline{v_{2_2}^4} + 4\overline{v_{1_2}}\,\overline{v_{2_2}}\,\overline{v_{2_0}} + 2\overline{v_{2_2}}\,\overline{v_{2_0}}\,\overline{v_{2_2}^2} + 2\overline{v_{1_0}v_{1_2}} + \overline{v_{1_2}}\,\overline{v_{2_0}^2} + \frac{1}{2}\overline{v_{2_2}^2v_{2_0}^2} \\
&+ \overline{v_{1_0}}\,\overline{v_{2_2}^2} + 2\overline{v_{1_0}}\,\overline{v_{2_2}}^2 + \overline{v_{2_2}}^2\overline{v_{2_0}^2}\big)\langle\phi_2^2\phi_0\rangle \\
&+ \big(\overline{v_{1_2}^2} + \overline{v_{1_2}}\,\overline{v_{1_2}^2} + \frac{1}{4}\overline{v_{2_2}^4} + 4\overline{v_{1_2}}\,\overline{v_{2_2}}\,\overline{v_{2_1}} + 2\overline{v_{2_2}}\,\overline{v_{2_1}}\,\overline{v_{2_2}^2} + 2\overline{v_{1_2}v_{1_1}} + \overline{v_{1_2}}\,\overline{v_{2_1}^2} + \frac{1}{2}\overline{v_{2_2}^2v_{2_1}^2} \\
&+ \overline{v_{1_1}}\,\overline{v_{2_2}^2} + 2\overline{v_{1_1}}\,\overline{v_{2_2}}^2 + \overline{v_{2_2}}^2\overline{v_{2_1}^2}\big)\langle\phi_2^2\phi_1\rangle \\
&+ \big(\overline{v_{1_2}^2} + \overline{v_{1_2}}\,\overline{v_{1_2}^2} + \frac{1}{4}\overline{v_{2_2}^4} + 2\overline{v_{1_2}}\,\overline{v_{2_2}}^2 + \overline{v_{2_2}}^2\overline{v_{2_2}^2}\big)\langle\phi_2^3\rangle \\
&+ \big(2\overline{v_{1_1}v_{1_0}} + 2\overline{v_{1_2}v_{1_1}} + 2\overline{v_{1_0}v_{1_2}} + \overline{v_{1_1}}\,\overline{v_{2_0}^2} + \overline{v_{1_2}}\,\overline{v_{2_1}^2} + \overline{v_{1_0}}\,\overline{v_{2_2}^2} \\
&+ \overline{v_{1_1}}\,\overline{v_{2_2}^2} + \overline{v_{1_2}}\,\overline{v_{2_0}^2} + \overline{v_{1_0}}\,\overline{v_{2_1}^2} + \frac{1}{2}(\overline{v_{2_1}^2v_{2_0}^2} + \overline{v_{2_0}^2v_{2_2}^2} + \overline{v_{2_2}^2v_{2_1}^2}) \\
&+ 4(\overline{v_{1_1}}\,\overline{v_{2_0}}\,\overline{v_{2_2}} + \overline{v_{1_2}}\,\overline{v_{2_1}}\,\overline{v_{2_0}} + \overline{v_{1_0}}\,\overline{v_{2_2}}\,\overline{v_{2_1}}) \\
&+ 2(\overline{v_{2_0}}\,\overline{v_{2_2}}\overline{v_{2_1}^2} + \overline{v_{2_1}}\,\overline{v_{2_0}}\,\overline{v_{2_2}^2} + \overline{v_{2_2}}\,\overline{v_{2_1}}\,\overline{v_{2_0}^2})\big)\langle\phi_0\phi_1\phi_2\rangle\Bigg).
\end{aligned}
$$

Appendix B. Haar wavelets.

We first need to repeat some properties of the Haar wavelets to give the proof of 11. By considering property two from Lemma 1, it was demonstrated in [26]:

Lemma 12. *[26]. A Haar wavelet basis fulfills:*

1. *For all $\hat{u}, \hat{w} \in \mathbb{R}^{K+1}$, $\mathcal{P}(\mathcal{P}(\hat{u})\hat{w}) = \mathcal{P}(\hat{u})\mathcal{P}(\hat{w})$ hold.*
2. *The mapping $\mathcal{T}_n : \mathbb{H}_0^+ \to \mathbb{H}_0^+, \quad \hat{u} \to \hat{u}^{*n} = \mathcal{P}^n(\hat{u})\hat{e}_1$ is bijective for all $n \in \mathbb{N}$, with inverse mapping $\mathcal{T}_n^{-1}(\hat{u}) = \mathcal{H}\mathrm{D}^{1/n}(\hat{u})\mathcal{H}^T\hat{e}_1$ and $\hat{e}_1 := (1, 0, \dots, 0, 0)^T \in \mathbb{R}^{K+1}$.*

It is shown in [49, Theorem 2.1] and [27, Theorem 2.1.] that condition (2.9) guarantees positive eigenvalues $\mathrm{D}(\hat{h})$ and thus the invertibility of $\mathcal{P}$.

Proof. (Theorem 11) The equation for $\hat{\alpha}$ is obtained from Lemma 12. Therefore we consider

$$\mathcal{T}_2(\hat{\alpha}) = \mathcal{P}^2(\hat{\alpha})\hat{e}_1 = \mathcal{P}(\hat{\alpha})\hat{\alpha} = \hat{\alpha} * \hat{\alpha} = \hat{h}$$

with the belonging inverse mapping $\mathcal{T}_2^{-1}(\hat{h}) = \mathcal{H}\mathrm{D}^{\frac{1}{2}}(\hat{h})\mathcal{H}^T\hat{e}_1$. We proceed with

$$
\begin{aligned}
\hat{\alpha} * \hat{\beta} = \hat{q} \Leftrightarrow \hat{\beta} = \mathcal{P}^{-1}(\hat{\alpha})\hat{q} &= \mathcal{H}\mathrm{D}^{-\frac{1}{2}}(\hat{h})\mathcal{H}^T\mathcal{P}(\hat{e}_1)\hat{q} = \mathcal{H}\mathrm{D}^{-\frac{1}{2}}(\hat{h})\mathcal{H}^T\mathcal{P}(\hat{q})\hat{e}_1 \\
&= \mathcal{H}\mathrm{D}^{-\frac{1}{2}}(\hat{h})\mathcal{H}^T\mathcal{H}\mathrm{D}(\hat{q})\mathcal{H}^T\hat{e}_1 = \mathcal{H}\mathrm{D}^{-\frac{1}{2}}(\hat{h})\mathrm{D}(\hat{q})\mathcal{H}^T\hat{e}_1.
\end{aligned}
$$

This results leads us to

$$\hat{\mathrm{v}} = \mathcal{P}^{-1}(\hat{\alpha})\hat{\beta} = \mathcal{H}\mathrm{D}^{-\frac{1}{2}}(\hat{h})\mathcal{H}^T\mathcal{H}\mathrm{D}^{-\frac{1}{2}}(\hat{h})\mathrm{D}(\hat{q})\mathcal{H}^T\hat{e}_1 = \mathcal{H}\mathrm{D}^{-1}(\hat{h})\mathrm{D}(\hat{q})\mathcal{H}^T\hat{e}_1.$$

■

Appendix C. Expansion to nonconstant bottom topographies. In the following, we explain shorty how to include bottom topography b inside the investigation and how to expand the numerical fluxes in this case. A straightforward way would be to extend as well b in the Galerkin product. The entropy/entropy variables changes afterwards similar to the purely deterministic case. In the purely deterministic case, we have for the entropy variables $v_1 = g(h+b) - \frac{1}{2}\mathsf{v}^2$ and $v_2 = \mathsf{v}$. Inserting the SG extension of b yields similar to (4.1), the entropy variables

$$(v_1^T, v_2^T) := \left(g(\hat{h}^T + \hat{b}^T) - \frac{1}{2}\hat{\mathsf{v}}^{2^T}, \hat{\mathsf{v}}^T \right)$$

where $\hat{b}$ defined through the Galerkin expansion. By following [43], we can directly adapt the approach for our SG-system. Therefore, an additional term containing water height h and bottom topography b needs to be added to the second numerical flux. By direct element-wise long calculations, we obtain in our case the numerical flux from Theorem 8

$$f_{2n}^{\text{num}} = \frac{1}{2g} \sum_{i,j=0}^{n} \Big(\overline{v_{1_j} v_{1_i}} + \overline{v_{1_j}}\; \overline{v_{2_i}^2} + \frac{1}{4}\overline{v_{2_j}^2 v_{2_i}^2} + 2\overline{v_{1_j}}\; \overline{v_{2_i}}\; \overline{v_{2_n}} + \overline{v_{2_i}}\; \overline{v_{2_n}}\overline{v_{2_j}^2} + \tag{C.1}$$

$$\frac{1}{2}\big(\overline{v_{1_i}} + \frac{1}{2}\overline{v_{1_i}^2} - g\overline{\hat{b}_i}\big)[[\hat{b}_j]]\Big)\langle \phi_i \phi_j \phi_n \rangle, \tag{C.2}$$

where the additional term delivers (C.2) which reads with filled-in mean and jumps as

$$\frac{1}{4}\left(v_{1_{j_+}} + v_{1_{j_-}} + \frac{1}{2}\left(v_{2_{i_+}}^2 + v_{2_{i_-}}^2 \right) - g\left(\hat{b}_{i_+} + \hat{b}_{i_-} \right) \right) \left(\hat{b}_{j_+} - \hat{b}_{j_-} \right).$$

It should be stressed out that this approach yields a consistent approximation to the deterministic case again. Testing these fluxes will be left for future work when more realistic benchmark tests will be considered and a comparison to other existing methods will be done. Additionally, the modeling aspect has also be considered in this case as well as the analytical properties similar to [24].

REFERENCES

[1] R. Abgrall, *A general framework to construct schemes satisfying additional conservation relations. application to entropy conservative and entropy dissipative schemes*, J. Comput. Phys., 372 (2018), pp. 640–666.

[2] R. Abgrall and S. Mishra, *Uncertainty qualification for hyperbolic systems of conservation laws*, in Handbook on numerical methods for hyperbolic problems. Applied and modern issues, Amsterdam: Elsevier/North Holland, 2017, pp. 507–544, https://doi.org/10.1016/bs.hna.2016.11.003.

[3] R. Abgrall, J. Nordström, P. Öffner, and S. Tokareva, *Analysis of the SBP-SAT stabilization for finite element methods part II: entropy stability*, Commun. Appl. Math. Comput., (2021), pp. 1–23.

[4] R. Abgrall, P. Öffner, and H. Ranocha, *Reinterpretation and extension of entropy correction terms for residual distribution and discontinuous Galerkin schemes: application to structure preserving discretization*, J. Comput. Phys., 453 (2022), p. 24, https://doi.org/10.1016/j.jcp.2022.110955. Id/No 110955.

[5] J. Bezanson, A. Edelman, S. Karpinski, and V. B. Shah, *Julia: a fresh approach to numerical computing*, SIAM Rev., 59 (2017), pp. 65–98, https://doi.org/10.1137/141000671, hdl.handle.net/1721.1/110125.

[6] R. H. Cameron and W. T. Martin, *The orthogonal development of non-linear functionals in series of Fourier-Hermite functionals*, Ann. Math. (2), 48 (1947), pp. 385–392, https://doi.org/10.2307/1969178.

[7] T. Chen and C.-W. Shu, *Entropy stable high order discontinuous Galerkin methods with suitable quadrature rules for hyperbolic conservation laws*, J. Comput. Phys., 345 (2017), pp. 427–461, https://doi.org/10.1016/j.jcp.2017.05.025, https://doi.org/10.1016/j.jcp.2017.05.025.

[8] T. Chen and C.-W. Shu, *Review of entropy stable discontinuous Galerkin methods for systems of conservation laws on unstructured simplex meshes*, CSIAM Trans. Appl. Math., 1 (2020), pp. 1–52.

[9] M. Ciallella, L. Micalizzi, P. Öffner, and D. Torlo, *An arbitrary high order and positivity preserving method for the shallow water equations*, Comput. Fluids, 247 (2022), p. 21, https://doi.org/10.1016/j.compfluid.2022.105630. Id/No 105630.

[10] M. Ciallella, D. Torlo, and M. Ricchiuto, *Arbitrary High Order WENO Finite Volume Scheme with Flux Globalization for Moving Equilibria Preservation*, arXiv preprint arXiv:2205.13315, (2022).

[11] B. Cockburn, G. E. Karniadakis, and C.-W. Shu, eds., *Discontinuous Galerkin methods. Theory, computation and applications. 1st international symposium on DGM, Newport, RI, USA, May 24–26, 1999*, vol. 11 of Lect. Notes Comput. Sci. Eng., Berlin: Springer, 2000.

[12] D. Dai, Y. Epshteyn, and A. Narayan, *Hyperbolicity-preserving and well-balanced stochastic Galerkin method for shallow water equations*, SIAM J. Sci. Comput., 43 (2021), pp. a929–a952, https://doi.org/10.1137/20M1360736.

[13] D. Dai, Y. Epshteyn, and A. Narayan, *Hyperbolicity-preserving and well-balanced stochastic Galerkin method for two-dimensional shallow water equations*, J. Comput. Phys., 452 (2022), p. 28, https://doi.org/10.1016/j.jcp.2021.110901. Id/No 110901.

[14] B. Després, G. Poëtte, and D. Lucor, *Robust uncertainty propagation in systems of conservation laws with the entropy closure method*, in Uncertainty quantification in computational fluid dynamics, Heidelberg: Springer, 2013, pp. 105–149, https://doi.org/10.1007/978-3-319-00885-1_3.

[15] B. Després, G. Poëtte, and D. Lucor, *Robust uncertainty propagation in systems of conservation laws with the entropy closure method*, in Uncertainty quantification in computational fluid dynamics, Heidelberg: Springer, 2013, pp. 105–149, https://doi.org/10.1007/978-3-319-00885-1_3.

[16] J. Dürrwächter, T. Kuhn, F. Meyer, L. Schlachter, and F. Schneider, *A hyperbolicity-preserving discontinuous stochastic Galerkin scheme for uncertain hyperbolic systems of equations*, J. Comput. Appl. Math., 370 (2020), p. 22, https://doi.org/10.1016/j.cam.2019.112602. Id/No 112602.

[17] T. C. Fisher, M. H. Carpenter, J. Nordström, N. K. Yamaleev, and C. Swanson, *Discretely conservative finite-difference formulations for nonlinear conservation laws in split form: Theory and boundary conditions*, Journal of Computational Physics, 234 (2013), pp. 353–375.

[18] U. S. Fjordholm, S. Mishra, and E. Tadmor, *Well-balanced and energy stable schemes for the shallow water equations with discontinuous topography*, J. Comput. Phys., 230 (2011), pp. 5587–5609, https://doi.org/10.1016/j.jcp.2011.03.042.

[19] U. S. Fjordholm, S. Mishra, and E. Tadmor, *Well-balanced and energy stable schemes for the shallow water equations with discontinuous topography*, J. Comput. Phys., 230 (2011), pp. 5587–5609, https://doi.org/10.1016/j.jcp.2011.03.042, citeseerx.ist.psu.edu/viewdoc/summary?doi=10.1.1.956.9625.

[20] E. Gaburro, P. Öffner, M. Ricchiuto, and D. Torlo, *High order entropy preserving ADER-DG schemes*, Appl. Math. Comput., 440 (2023), p. 21, https://doi.org/10.1016/j.amc.2022.127644. Id/No 127644.

[21] G. J. Gassner, *A skew-symmetric discontinuous galerkin spectral element discretization and its relation to sbp-sat finite difference methods*, SIAM Journal on Scientific Computing, 35 (2013), pp. A1233–A1253.

[22] G. J. Gassner, A. R. Winters, and D. A. Kopriva, *Split form nodal discontinuous galerkin schemes with summation-by-parts property for the compressible euler equations*, Journal of Computational Physics, 327 (2016), pp. 39–66.

[23] G. J. Gassner, A. R. Winters, and D. A. Kopriva, *A well balanced and entropy conservative discontinuous Galerkin spectral element method for the shallow water equations*, Appl. Math. Comput., 272 (2016), pp. 291–308, https://doi.org/10.1016/j.amc.2015.07.014, urn.kb.se/resolve?urn=urn:nbn:se:liu:diva-156863.

[24] S. Gerster and M. Herty, *Entropies and symmetrization of hyperbolic stochastic Galerkin formulations*, Commun. Comput. Phys., 27 (2020), pp. 639–671, https://doi.org/10.4208/cicp.OA-2019-0047.

[25] S. Gerster, M. Herty, and A. Sikstel, *Hyperbolic stochastic Galerkin formulation for the p-system*, J. Comput. Phys., 395 (2019), pp. 186–204, https://doi.org/10.1016/j.jcp.2019.05.049.

[26] S. Gerster, A. Sikstel, and G. Visconti, *Haar-type stochastic galerkin formulations for hyperbolic systems with lipschitz continuous flux function*, 2022.

[27] D. Gottlieb and D. Xiu, *Galerkin method for wave equations with uncertain coefficients*, Commun. Comput. Phys., 3 (2008), pp. 505–518.

[28] M. Herty, A. Kolb, and S. Müller, *Higher-dimensional deterministic formulation of hyperbolic conservation laws with uncertain initial data*, Institut für Geometrie und Praktische Mathematik, RWTH Aachen, 2021.

[29] M. Herty, A. Kolb, and S. Müller, *Multiresolution-analysis for stochastic hyperbolic conservation laws*, arXiv preprint arXiv:2203.11534, (2022).

[30] D. Kuzmin, H. Hajduk, and A. Rupp, *Limiter-based entropy stabilization of semi-discrete and fully discrete schemes for nonlinear hyperbolic problems*, Comput. Methods Appl. Mech. Eng., 389 (2022), p. 28, https://doi.org/10.1016/j.cma.2021.114428. Id/No 114428.

[31] P. G. LeFloch, J.-M. Mercier, and C. Rohde, *Fully discrete, entropy conservative schemes of arbitrary order*, SIAM Journal on Numerical Analysis, 40 (2002), pp. 1968–1992.

[32] Y. Mantri, P. Öffner, and M. Ricchiuto, *Fully well balanced entropy controlled dgsem for shallow water flows: global flux quadrature and cell entropy correction*, arXiv preprint arXiv:2212.11931, (2022).

[33] A. Meister and S. Ortleb, *On unconditionally positive implicit time integration for the DG scheme applied to shallow water flows*, International Journal for Numerical Methods in Fluids, 76 (2014), pp. 69–94.

[34] S. Michel, D. Torlo, M. Ricchiuto, and R. Abgrall, *Spectral analysis of high order continuous FEM for hyperbolic PDEs on triangular meshes: influence of approximation, stabilization, and time-stepping*, Journal of Scientific Computing, 94 (2023), p. 49, https://doi.org/10.1007/s10915-022-02087-0.

[35] S. Mishra, N. H. Risebro, C. Schwab, and S. Tokareva, *Numerical solution of scalar conservation laws with random flux functions*, SIAM/ASA J. Uncertain. Quantif., 4 (2016), pp. 552–591, https://doi.org/10.1137/120896967,

hdl.handle.net/20.500.11850/154924.

[36] P. Öffner, *Approximation and Stability Properties of Numerical Methods for Hyperbolic Conservation Laws*, Habilitation, University Zurich, 2020.

[37] P. Öffner, J. Glaubitz, and H. Ranocha, *Stability of correction procedure via reconstruction with summation-by-parts operators for Burgers' equation using a polynomial chaos approach*, ESAIM, Math. Model. Numer. Anal., 52 (2018), pp. 2215–2245, https://doi.org/10.1051/m2an/2018072.

[38] P. Öffner, H. Ranocha, and T. Sonar, *Correction procedure via reconstruction using summation-by-parts operators*, in Theory, numerics and applications of hyperbolic problems II, Aachen, Germany, August 2016, Cham: Springer, 2018, pp. 491–501, https://doi.org/10.1007/978-3-319-91548-7_37.

[39] M. Petrella, S. Tokareva, and E. F. Toro, *Uncertainty quantification methodology for hyperbolic systems with application to blood flow in arteries*, J. Comput. Phys., 386 (2019), pp. 405–427, https://doi.org/10.1016/j.jcp.2019.02.013, www.osti.gov/biblio/1499352.

[40] M. P. Pettersson, G. Iaccarino, and J. Nordström, *Polynomial chaos methods for hyperbolic partial differential equations. Numerical techniques for fluid dynamics problems in the presence of uncertainties*, Math. Eng. (Cham), Cham: Springer, 2015, https://doi.org/10.1007/978-3-319-10714-1.

[41] P. Pettersson, G. Iaccarino, and J. Nordström, *A stochastic Galerkin method for the Euler equations with Roe variable transformation*, J. Comput. Phys., 257 (2014), pp. 481–500, https://doi.org/10.1016/j.jcp.2013.10.011.

[42] G. Poëtte, B. Després, and D. Lucor, *Uncertainty quantification for systems of conservation laws*, J. Comput. Phys., 228 (2009), pp. 2443–2467, https://doi.org/10.1016/j.jcp.2008.12.018.

[43] H. Ranocha, *Shallow water equations: split-form, entropy stable, well-balanced, and positivity preserving numerical methods*, GEM. Int. J. Geomath., 8 (2017), pp. 85–133, https://doi.org/10.1007/s13137-016-0089-9.

[44] H. Ranocha, M. Schlottke-Lakemper, A. R. Winters, E. Faulhaber, J. Chan, and G. Gassner, *Adaptive numerical simulations with Trixi.jl: A case study of Julia for scientific computing*, Proceedings of the JuliaCon Conferences, 1 (2022), p. 77, https://arxiv.org/abs/2108.06476.

[45] M. Ricchiuto, R. Abgrall, and H. Deconinck, *Application of conservative residual distribution schemes to the solution of the shallow water equations on unstructured meshes*, Journal of Computational Physics, 222 (2007), pp. 287–331.

[46] L. Schlachter and F. Schneider, *A hyperbolicity-preserving stochastic Galerkin approximation for uncertain hyperbolic systems of equations*, J. Comput. Phys., 375 (2018), pp. 80–98, https://doi.org/10.1016/j.jcp.2018.07.026.

[47] M. Schlottke-Lakemper, G. J. Gassner, H. Ranocha, A. R. Winters, and J. Chan, *Trixi.jl: Adaptive high-order numerical simulations of hyperbolic PDEs in Julia.* https://github.com/trixi-framework/Trixi.jl, 09 2021.

[48] C. Schwab and S. Tokareva, *High order approximation of probabilistic shock profiles in hyperbolic conservation laws with uncertain initial data*, ESAIM, Math. Model. Numer. Anal., 47 (2013), pp. 807–835, https://doi.org/10.1051/m2an/2012060.

[49] B. E. Sonday, R. D. Berry, H. N. Najm, and B. J. Debusschere, *Eigenvalues of the Jacobian of a Galerkin-projected uncertain ODE system*, SIAM J. Sci. Comput., 33 (2011), pp. 1212–1233, https://doi.org/10.1137/100785922.

[50] E. Tadmor, *The numerical viscosity of entropy stable schemes for systems of conservation laws. I*, Math. Comput., 49 (1987), pp. 91–103, https://doi.org/10.2307/2008251.

[51] E. Tadmor, *Entropy stability theory for difference approximations of nonlinear conservation laws and related time-dependent problems*, Acta Numerica, 12 (2003), pp. 451–512, https://doi.org/10.1017/S0962492902000156.

[52] S. Tokareva, C. Schwab, and S. Mishra, *High order SFV and mixed SDG/FV methods for the uncertainty quantification in multidimensional conservation laws*, in High order nonlinear numerical schemes for evolutionary PDEs. Proceedings of the European workshop HONOM 2013, Bordeaux, France, March 18–22, 2013, Cham: Springer, 2014, pp. 109–133, https://doi.org/10.1007/978-3-319-05455-1_7.

[53] X. Wen, W. S. Don, Z. Gao, and Y. Xing, *Entropy stable and well-balanced discontinuous Galerkin methods for the nonlinear shallow water equations*, J. Sci. Comput., 83 (2020), p. 32, https://doi.org/10.1007/s10915-020-01248-3. Id/No 66.

[54] N. Wiener, *The homogeneous chaos*, Am. J. Math., 60 (1938), pp. 897–936, https://doi.org/10.2307/2371268.

[55] A. R. Winters and G. J. Gassner, *A comparison of two entropy stable discontinuous Galerkin spectral element approximations for the shallow water equations with non-constant topography*, J. Comput. Phys., 301 (2015), pp. 357–376, https://doi.org/10.1016/j.jcp.2015.08.034, urn.kb.se/resolve?urn=urn:nbn:se:liu:diva-156867.

[56] X. Wu, E. J. Kubatko, and J. Chan, *High-order entropy stable discontinuous Galerkin methods for the shallow water equations: curved triangular meshes and GPU acceleration*, Comput. Math. Appl., 82 (2021), pp. 179–199, https://doi.org/10.1016/j.camwa.2020.11.006.

[57] T. Xiao, J. Kusch, J. Koellermeier, and M. Frank, *A Flux Reconstruction Stochastic Galerkin Scheme for Hyperbolic Conservation Laws*, J. Sci. Comput., 95 (2023), https://doi.org/10.1007/s10915-023-02143-3. Id/No 18.
[58] Y. Xing and C.-W. Shu, *A survey of high order schemes for the shallow water equations*, J. Math. Study, 47 (2014), pp. 221–249.
[59] D. Xiu and J. S. Hesthaven, *High-order collocation methods for differential equations with random inputs*, SIAM J. Sci. Comput., 27 (2005), pp. 1118–1139, https://doi.org/10.1137/040615201, infoscience.epfl.ch/record/190489.
[60] D. Xiu and G. E. Karniadakis, *The Wiener–Askey polynomial chaos for stochastic differential equations*, SIAM J. Sci. Comput., 24 (2002), pp. 619–644, https://doi.org/10.1137/S1064827501387826.
[61] X. Zhong and C.-W. Shu, *Entropy stable Galerkin methods with suitable quadrature rules for hyperbolic systems with random inputs*, J. Sci. Comput., 92 (2022), p. 30, https://doi.org/10.1007/s10915-022-01866-z. Id/No 14.